Topology of Lagrangian immersions and submanifolds

by

Christian Hilaire

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Author..

Department of Mathematics
May 7, 2010

Certified by..

Katrin Wehrheim
Assistant Professor of Mathematics
Thesis Supervisor

Accepted by..

Bjorn Poonen
Chairman, Department Committee on Graduate Students

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Abstract

In this thesis, we look at some of the first topological results for Lagrangian immersions and embeddings. In particular, we state and consider some applications of the h-principle of Gromov which gives a homotopy classification of Lagrangian immersions. We outline a proof of Matsushima's theorem which states that there is no Lagrangian embedding $\mathbb{S}^n \to (\mathbb{C}^n, \omega)$ if $n \neq 1, 3$ and ω is any symplectic form on $\mathbb{C}^n$. We define the notions of the Malsov class and of monotone Lagrangian immersions or embeddings and we give some examples.

Thesis Supervisor: Katrin Wehrheim
Title: Assistant Professor of Mathematics

1 Basic symplectic geometry

In this section, we list some basic notions of symplectic geometry that we will need for the rest of the paper. The proofs can be found in any introductory textbook in symplectic geometry such as [13].

1.1 Symplectic vector spaces

Definition 1.1. A skew-symmetric bilinear form ω on a real vector space V is said to be nondegenerate if

$$\omega(u,v) = 0 \quad \forall u \in V \Rightarrow v = 0.$$

The vector space V equipped with the nondegenerate form ω is called a symplectic vector space.

A skew-symmetric bilinear form ω defines a linear map

$$\begin{aligned} \omega^{\#} : V &\to V^* \\ v &\to \omega(v, .). \end{aligned}$$

If V is a finite dimensional vector space, the nondegeneracy condition is equivalent to the condition of $\omega^{\#}$ being one-to-one. For now on, all vector spaces will be assumed to be finite-dimensional.

Definition 1.2. Suppose (V_0, ω_0) and (V_1, ω_1) are symplectic vector spaces. An invertible linear map $T : V_0 \to V_1$ is called a symplectomorphism if $T^*\omega_1 = \omega_0$.

Example 1.3. Let V be the standard vector space $\mathbb{R}^{2n}$ with coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$. If we set $\omega_0 = \sum dx_i \wedge dy_i$, then $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic vector space. It is called the standard symplectic vector space.

A symplectomorphism $T : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ is represented by a matrix A satisfying $A^\top J_0 A = J_0$ where

$$J_0 = \begin{bmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{bmatrix}.$$

and $I_{n\times n}$ is the $n \times n$ identity matrix.

Definition 1.4. The set of $2n \times 2n$ matrices A which satisfy

$$A^{\top} J_0 A = J_0$$

where J_0 is given above form a Lie group called the symplectic group. It will be denoted by $Sp(2n)$.

$Sp(2n)$ is a subgroup of the general linear group $Gl(2n, \mathbb{R})$ since $Sp(2n)$ consists of invertible linear maps.

Example 1.5. Let W be any finite-dimensional real vector space and let W^* be the corresponding dual vector space. Consider the skew-symmetric bilinear form defined on $V = W \oplus W^*$ by

$$\begin{aligned} \omega : V \times V &\rightarrow \mathbb{R} \\ ((u_1, \alpha_1); (u_2, \alpha_2)) &\rightarrow \alpha_1(u_2) - \alpha_2(u_1). \end{aligned}$$

One can check that the alternating form ω is nondegenerate. Thus, it gives V the structure of a symplectic vector space.

Any symplectic vector space is symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. This is due to the following theorem.

Theorem 1.6. *Let (V, ω) be a finite dimensional symplectic vector space. Then*

1. *V is even dimensional.*
2. *there is a basis $e_1, \dots, e_n, f_1, \dots, f_n$ such that $n = \frac{\dim V}{2}$ and*

$$\begin{aligned} \omega(e_i, f_j) &= \delta_{ij} \\ \omega(e_i, e_j) = \omega(f_i, f_j) &= 0 \end{aligned}$$

where δ_{ij} is the Kronecker delta function.

Hence, $\omega = \Sigma e_i^* \wedge f_i^*$ where $\{e_1^*, \dots, e_n^*, f_1^*, \dots, f_n^*\}$ is the dual basis. The basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is called a symplectic basis.

It follows that any two symplectic vector spaces of the same dimension are symplectomorphic.

Definition 1.7. Let (V, ω) be a symplectic vector space and let $U \subset V$ be a linear subspace. The symplectic perpendicular to U in V is the space

$$U^\omega := \{v \in V | \omega(u, v) = 0, \forall u \in U\}.$$

The following properties are easy to verify. Given $U \subset V$

1. $(U^\omega)^\omega = U$
2. $\dim U^\omega + \dim U = \dim V$

Definition 1.8. A subspace U of (V, ω) is said to be Lagrangian if $U^\omega = U$. Equivalently, U is Lagrangian if

$$\dim U = \frac{1}{2} \dim V \text{ and } \omega(u, v) = 0 \ \forall u, v \in V.$$

The following lemma states that symplectomorphisms of symplectic vector spaces preserve Lagrangian subspaces.

Lemma 1.9. *If $T : (V_0, \omega_0) \to (V_1, \omega_1)$ is a symplectomorphism and $L \subset V_0$ a Lagrangian subspace, then $T(L) \subset V_1$ is a Lagrangian subspace.*

Example 1.10.

1. Any one-dimensional subspace of a two-dimensional symplectic vector space is Lagrangian.
2. Consider the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ from Example 1.3. Then the subspace $\mathbb{R}^n \times \{0\}$ given by setting $y_1 = \cdots = y_n = 0$ is a Lagrangian subspace.
3. If we represent a vector of $\mathbb{R}^{2n}$ by $(\vec{x}, \vec{y})$ where $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$, then ω_0 can be expressed as

$$\omega((\vec{x}_1, \vec{y}_1); (\vec{x}_2, \vec{y}_2)) = <\vec{x}_1, \vec{y}_2> - <\vec{x}_2, \vec{y}_1>$$

where $< ., . >$ is the standard inner product on $\mathbb{R}^n$. Let A be an $n \times n$ matrix. Then the subspace $\{(\vec{x}, A\vec{x}) | \vec{x} \in \mathbb{R}^n\}$ is Lagrangian if and only if A is symmetric.

We identify the complex vector space $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the correspondence

$$(z_1, \cdots, z_n) \to (x_1, \cdots, x_n, y_1, \cdots, y_n) \tag{1.1}$$

where $z_j = x_j + iy_j$. Let $(.,.)$ be the standard Hermitian inner product on $\mathbb{C}^n$. Then ω_0 is given by

$$\omega_0(z, w) = -Im\left((z, w)\right) \text{ for } z, w \in \mathbb{C}^n.$$

In fact, we can write $(.,.)$ as

$$(.,.) = < .,. > -i\omega_0(.,)$$

where $< .,. >$ is the standard Euclidean inner product on $\mathbb{R}^{2n}$. If we view multiplication by i as a real linear isomorphism of $\mathbb{R}^{2n}$, it satisfies $i^2 = -Id$. The concept of the operator i generalizes to arbitrary even-dimensional vector spaces.

Definition 1.11. A complex structure on a real vector space V is an isomorphism $J : V \to V$ such that $J^2 = -Id$.

A complex structure on a real vector space V gives it the structure of a complex vector space via

$$(a + ib) \cdot v = av + bJv$$

where $(a + ib) \in \mathbb{C}$ and $v \in V$. The existence of a complex structure implies that the dimension of V is even. In fact, it can be shown that there exists $e_1, \cdots, e_n \in V$ such that $e_1, \cdots, e_n, Je_1, \cdots, Je_n$ is a basis of V. Conversely, given a basis $e_1, \cdots, e_n, f_1, \cdots, f_n$, we get a complex structure $J : V \to V$ defined by $Je_i = f_i$ and $Jf_i = -e_i$ for $i = 1 \cdots, n$. So the space of complex structures on an even-dimensional vector space is non-empty.

Definition 1.12. Let J be a complex structure on a symplectic vector space (V, ω). J is said to be compatible with ω (or ω-compatible) if

1. $\omega(u, Ju) > 0$ and
2. $\omega(Ju, Jv) = \omega(u, v)$

for all $u, v \in V$. A ω-compatible complex structure J on (V, ω) defines an inner product g on (V, ω) given by

$$g(u, v) = \omega(u, Jv)$$

We also get a hermitian inner product on the complex vector space (V, J) defined by

$$h(u, v) = g(u, v) - i\omega(u, v).$$

By the following lemma, a ω-compatible complex structure always exists.

Lemma 1.13. *Let (V, ω) be a symplectic vector space. Then there exists a complex structure J on V compatible with ω.*

ω-compatible complex structures give us a new way of defining Lagrangian subspaces.

Lemma 1.14. *Let J be a ω-compatible complex structure on (V, ω). Then $L \subset V$ is Lagrangian if and only if $J(L) = L^{\perp}$, where the orthogonality is with respect to the inner product $g(.,.) = \omega(., J.)$.*

1.2 Symplectic vector bundles

Definition 1.15. A symplectic vector bundle over a manifold M is a vector bunlde $E \xrightarrow{\pi} M$ such that, for each $x \in M$, the fiber E_x is equipped with a symplectic structure ω_x which varies continuously with x. The symplectic vector bundle is said to be smooth if the underlying vector bundle is smooth and ω varies smoothly with $x \in M$.

The rank of a symplectic vector bundle is necessarily even. It can be shown that a symplectic vector bundle is locally trivial: for each $x \in M$, there exists a neighborhood $U \subset M$ and a commutative diagram

$$\begin{array}{ccc} (E_{|U}, \omega) & \xrightarrow{\Phi} & U \times (\mathbb{R}^{2n}, \omega_0) \\ \pi \downarrow & & \downarrow p_1 \\ U & = & U \end{array}$$

where, for each $x \in U$, $\Phi_x : E_x \to (\mathbb{R}^{2n}, \omega_0)$ is a symplectomorphism. The structure group of a symplectic vector bundle is $Sp(2n)$. Furthermore, every symplectic vector bundle has the structure of a unitary vector bundle.

Lemma 1.16. *Suppose $(E, \omega) \to M$ is a symplectic vector bundle. Then there is a compatible complex structure J on E and a hermitian inner product h on the complex bundle $(E, J) \to M$ given by*

$$h_x(u, v) = g_x(u, v) - i\omega_x(u, v)$$

where $u, v \in E_x$, $x \in M$ and g is an inner product defined by $g(.,.) = \omega(., J.)$. We say that $(E, J, h) \to M$ is a compatible unitary structure.

This result can be seen as an extension of Lemma 1.13. It is due to the fact that the unitary group $U(n)$ is a maximal compact subgroup of $Sp(2n)$. So the structure group of a symplectic vector bundle can be reduced to $U(n)$. In particular, the classifiyng space of symplectic vector bundles $BSp(2n)$ is homotopy equivalent to the classifying space of unitary vector bundles $BU(n)$.

Definition 1.17. A Lagrangian subbundle of a symplectic vector bundle $(E, \omega) \to M$ is a subbundle $L \to M$ such that L_x is a Lagrangian subspace of E_x for each $x \in M$.

Definition 1.18. Suppose J is a complex structure on a vector space V. Then a subspace L is said to be totally real if it has dimension $\dim L = \frac{\dim V}{2}$ and $J(L) \cap L = \{0\}$. A subbundle $L \to M$ of a vector bundle $E \to M$ with complex structure J is totally real if, for each $x \in M$, L_x is a totally real subspace of V_x.

Example 1.19. Recall that a Lagrangian subspace L of a symplectic vector space (V, ω) satisfies $J(L) = L^\perp$ where J is ω-compatible and the orthogonality is with respect to the inner product given by $g(.,.) = \omega(., J.)$. In particular, $J(L) \cap L = \{0\}$. Hence, a Lagrangian subspace of a symplectic vector space (V, ω) is totally real with respect to any ω-compatible complex structure.

Note that every symplectic vector space admits a Lagrangian subspace but a symplectic vector bundle does not necessarily admit a Lagrangian subbundle. We have the following result.

Lemma 1.20. *Suppose $E \to M$ is a vector bundle with complex structure J. Then $L \to M$ is a totally real subbundle if and only if the inclusion bundle map $\iota : L \to E$ induces a complex vector bundle isomorphism*

$$\iota^{\mathbb{C}} : L \otimes_{\mathbb{R}} \mathbb{C} \to (E, J).$$

Recall that $L \otimes_{\mathbb{R}} \mathbb{C} \to M$ is the complexification of the bundle $L \to M$.

Proof. The complex bundle map

$$\begin{tikzcd} L \otimes_{\mathbb{R}} \mathbb{C} \arrow[rr, "\iota^{\mathbb{C}}"] \arrow[dr] & & (E, J) \arrow[dl] \\ & M & \end{tikzcd}$$

is given by

$$(x, v \otimes (a + ib)) \to (x, av + bJ_x(v)).$$

This is a complex vector bundle isomorphism if and only if

1. $J_x(L_x) \cap L_x = \{0\}$ for each $x \in M$. (The bundle map is injective.)
2. $rank(L) = \frac{1}{2} rank(E)$. (The bundle map is surjective.)

So it is an isomorphism if and only if L is a totaly real subbundle of (E, ω). □

Corollary 1.21. *Suppose $(E, \omega) \to M$ is a symplectic vector bundle and $(E, J, h) \to M$ is a compatible unitary structure. If there exists a Lagrangian subbundle $L \to M$, then its complexification is isomorphic to $(E, J) \to M$.*

Proof. The Lagrangian subbundle is totally real with respect to the complex structure J. Hence, we can apply Lemma 1.20. □

1.3 Symplectic manifolds

A very important example of a symplectic vector bundle $(E, \omega) \to M$ is the case where the total space E is the tangent bundle TM of M and ω is a closed 2-form on M.

Definition 1.22. Let M be a 2n-dimensional smooth manifold. A 2-form $\omega \in \Omega^2(M)$ is nondegenerate if and only if for every $x \in M$, the bilinear form ω_x on $T_x M$ is nondegenerate. A symplectic manifold is a pair (M, ω) where M is a manifold and ω is a nondegenerate closed 2-form on M.

Example 1.23. Let $M = \mathbb{R}^{2n}$ with coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$. The 2-form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ gives $\mathbb{R}^{2n}$ the structure of a symplectic manifold. This corresponds to the standard symplectic vector space from Example 1.3. If we use the identification $\mathbb{R}^{2n} = \mathbb{C}^n$ given by Equation 1.1, the symplectic form is given by $\omega_0 = \frac{i}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$.

Note that the symplectic form ω_0 from Example 1.23 is exact since every closed form on $\mathbb{R}^{2n}$ is exact. If (M, ω) is a closed symplectic manifold, then ω cannot be an exact form. Indeed, the nondegeneracy condition on ω is equivalent to the condition that the top-form $\omega^{\wedge n} \in \Omega^{2n}(M)$ is a volume form on M. In particular, if M is a closed manifold and $\omega = d\beta$, then this implies that $\omega^{\wedge n} = d(\omega^{(n-1)} \wedge \beta)$. Hence, $\int_M \omega^{\wedge n} = 0$ by Stokes' theorem. We get a contradiction.

Definition 1.24. Two symplectic manifolds (M_0, ω_0) and (M_1, ω_1) are symplectomorphic if there exists a diffeomorphism $\phi : M_0 \to M_1$ such that $\phi^*\omega_1 = \omega_0$. The map ϕ is called a symplectomorphism.

Definition 1.25. Suppose (M, ω) is symplectic manifold. Then a vector field X on M is said to be symplectic if its flow $\phi_t : M \to M$ preserves the symplectic form ω.

Suppose X is a symplectic vector field on a symplectic manifold (M, ω). Let $\phi_t : (M, \omega) \to (M, \omega)$ be the corresponding one-parameter group of symplectomorphisms. Then

$$\begin{aligned} \mathcal{L}_X\omega &= \frac{d}{dt}\phi_t^*\omega|_{t=0} \\ \mathcal{L}_X\omega &= \frac{d}{dt}\omega|_{t=0} \\ \mathcal{L}_X\omega &= 0 \end{aligned} \tag{1.2}$$

Therefore, a vector field X on (M, ω) is symplectic if and only if it satisfies the equation above. If we use the fact that ω is a closed 2-form and the identity

$$\mathcal{L}_X = d\iota_X + \iota_X d$$

we can reduce Equation 1.2 to

$$d\iota_X\omega = 0$$

Lemma 1.26. *A vector field X on a symplectic manifold (M, ω) is symplectic if and only if the 1-form $\iota_X \omega$ is closed.*

We can construct symplectic vector fields on a symplectic manifold as follows. For every element x of a symplectic manifold (M, ω), the map

$$(\omega_x)^\# : T_x M \to T_x^* M$$

is an isomorphism. This induces a one-to-one correspondence between 1-forms and vector fields. Given a function $f \in C^\infty(M)$, the differential df is a 1-form. We get the corresponding vector field $X_f(x) := (\omega_x^\#)^{-1}(d_x f)$. Equivalently, the vector field X_f is defined by $\iota_{X_f}\omega = df$. This vector field is indeed symplectic since the 1-form $\iota_{X_f}\omega = df$ is exact and, hence, closed.

Definition 1.27. Suppose (M, ω) is a symplectic manifold and $f \in C^\infty(M)$. The vector field X_f given by

$$\iota_{X_f}\omega = df$$

is called the Hamiltonian vector field of f.

If X is the Hamiltonian vector field of a function f, we also say that f is the Hamiltonian of X. Note that the Hamiltonian of a Hamiltonian vector field is not unique. If M is connected, any two Hamiltonian functions of a Hamiltonian vector field differ by a constant.

Example 1.28. Consider the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ from Example 1.23. For any $f \in C^\infty(\mathbb{R}^{2n})$, the Hamiltonian vector field is

$$X_f = \sum_{i=1}^{n} \left(-\frac{\partial f}{\partial x_i}\frac{\partial}{\partial y_i} + \frac{\partial f}{\partial y_i}\frac{\partial}{\partial x_i} \right)$$

It turns out that every symplectic vector field on $(\mathbb{R}^{2n}, \omega_0)$ is a Hamiltonian vector field. Indeed, if X is a symplectic vector field, the 1-form $\iota_X \omega_0$ is closed by Lemma 1.26. Since every closed form on $\mathbb{R}^{2n}$ is exact, we must have $\iota_X \omega_0 = df$ for some $f \in C^\infty(\mathbb{R}^{2n})$.
For instance, the constant vector field $X = v \in \mathbb{R}^{2n}$ generates the one-parameter group of translations

$$\begin{array}{rcl} \phi_t : \mathbb{R}^{2n} & \to & \mathbb{R}^{2n} \\ p & \to & p + tv. \end{array}$$

One can check that any translation map is a symplectomorphism. So, the vector field X is Hamiltonian. If $(c_1, \cdots, c_n, d_1, \cdots, d_n)$ are the coordinates of the vector v with respect to the basis $(\frac{\partial}{\partial x_1} \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1} \cdots, \frac{\partial}{\partial y_n})$, then a corresponding Hamiltonian function is given by

$$\begin{aligned} f : \mathbb{R}^{2n} &\to \mathbb{R} \\ p &\to \sum_{i=1}^{n} (-d_i x_i(p) + c_i x_i(p)). \end{aligned}$$

Example 1.29. For any oriented surface, a symplectic form is simply an area form. Hence, a two-dimensional symplectic manifold is simply an oriented surface equipped with an area form. For instance, consider the unit 2-sphere in $\mathbb{R}^3$.

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 | \, x^2 + y^2 + z^2 = 1\}$$

We define a symplectic form on $\mathbb{S}^2$ as follows. For any $p \in \mathbb{S}^2$, the tangent space is given by

$$T_p\mathbb{S}^2 = \{u \in \mathbb{R}^3 | \, < u, p >= 0\}$$

As before, $< ., . >$ denotes the standard Euclidean inner product on $\mathbb{R}^3$. The symplectic form ω is given by

$$\begin{aligned} T_p\mathbb{S}^2 \times T_p\mathbb{S}^2 &\to \mathbb{R} \\ (u, v) &\to < p, u \times v > \end{aligned}$$

for every $p \in \mathbb{S}^2$. Here, $u \times v$ denotes the standard cross product operation in $\mathbb{R}^3$. One can check that ω is indeed a smooth non-degenerate 2-form. In fact, if we use cylindrical coordinates (r, θ, z), then the symplectic form is locally given by $\omega = dz \wedge d\theta$.

Theorem 1.6 states that the symplectic bilinear form of a vector space can be expressed in a standard form. Thus, one deduces that symplectic vector spaces of the same dimension are symplectomorphic. In the case of symplectic manifolds, we can only claim that symplectic manifolds of a given dimension are locally equivalent.

Theorem 1.30 (Darboux's theorem). *Suppose (M, ω) is a symplectic manifold and $p \in M$. There exist coordinates*

$$(x_1, \cdots, x_n, y_1, \cdots, y_n) : U \to \mathbb{R}^{2n}$$

defined on a neighborhood $U \subset M$ of p such that $\omega = \sum dx_i \wedge dy_i$.

Hence, all symplectic manifolds of a given dimension are locally symplectomorphic. The coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are called symplectic coordinates.

Example 1.31 (Cotangent bundles and the Liouville form)**.** The cotangent bundle T^*M of a manifold M is an even-dimensional manifold. It has a natural symplectic structure defined as follows.

Let $\pi : T^*M \to M$ be the projection map. Consider $\eta \in T^*M$ and $p = \pi(\eta)$. Hence, $\eta \in T_x^*M$ and

$$\pi_* : T_\eta(T^*M) \to T_pM$$

We define a 1-form $\alpha \in \Omega^1(T^*M)$ by

$$\alpha_\eta(v) = \eta(\pi_*(v)).$$

The 1-form α is smooth. In fact, given a coordinate chart $(U, x_1, \ldots, x_n)$ for M, we have the associated coordinate chart $(T^*U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ where, for $\eta \in T_p^*U$, $(\xi_1(\eta), \cdots, \xi_n(\eta))$ are the coordinates with respect to the basis $(dx_1|_p, \cdots, dx_n|_p)$. Then, locally,

$$\begin{aligned} \alpha &= \sum \xi_i dx_i \\ d\alpha &= \sum d\xi_i \wedge dx_i. \end{aligned}$$

The 1-form $-d\alpha$ is closed and nondegenerate. Thus, it defines a symplectic form on T^*M. Note that the local coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ are symplectic coordinates. The 1-form α is called the tautological form or Liouville form.

Example 1.32. Consider the complex projective space $\mathbb{C}P^n$ with homogeneous coordinates $[w_0 : \cdots : w_n]$. The space $\mathbb{C}P^n$ is covered by the open sets

$$U_j = \{[w_0 : \cdots : w_n] \in \mathbb{C}P^n \mid w_j \neq 0\}$$

where $j = 0, \cdots, n$. Recall that $\mathbb{C}P^n$ has the structure of a complex manifold with coordinate charts

$$\psi_j : U_j \to \mathbb{C}^n$$

given by

$$[w_0 : \cdots : w_n] \quad \to \quad (\frac{w_0}{w_j}, \cdots, \frac{w_{j-1}}{w_j}, \frac{w_{j+1}}{w_j}, \cdots, \frac{w_n}{w_j}) \tag{1.3}$$

for $j = 0, \cdots, n$. As a real manifold, $\mathbb{C}P^n$ has a symplectic structure ω which is locally given by

$$\omega = \frac{i}{2}\partial\overline{\partial}log(1 + \sum_{k=1}^{n} |z_k|^2)$$

on each set U_j. Here, $(z_1, \cdots, z_n)$ are the local coordinates on U_j given by 1.3 and ∂, $\overline{\partial}$ are the corresponding complex differentials.

Definition 1.33. Suppose (M, ω) is a symplectic manifold and L a smooth manifold. An immersion $f : L \to (M, \omega)$ is Lagrangian if, for each $x \in L$, the differential $d_x f$ maps $T_x L$ to a Lagrangian subspace of $(T_{f(x)}M, \omega_{f(x)})$. Equivalently, the immersion is Lagrangian if and only if $\dim L = \frac{\dim M}{2}$ and $f^*\omega = 0$. If f is an embedding, $f(L)$ is called a Lagrangian submanifold of (M, ω). For simplicity, we will also say that L is a Lagrangian submanifold of (M, ω) in the latter case.

In Lemma 1.9, it was stated that symplectomorphisms of symplectic vector spaces map Lagrangian subspaces to Lagrangian subspaces. An equivalent result holds for symplectomorphisms of symplectic manifolds.

Lemma 1.34. *Suppose $\phi : (M_0, \omega_0) \to (M_1, \omega_1)$ is a symplectomorphism and $f : L \to (M_0, \omega_0)$ a Lagrangian immersion (embedding). Then the composite map $\phi \circ f : L \to (M_1, \omega_1)$ is a Lagrangian immersion (embedding).*

Example 1.35. Any one-dimensional submanifold of a 2-dimensional symplectic manifold is Lagrangian.

Example 1.36. Consider the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ from Example 1.23. The submanifold

$$\mathbb{R}^n \times \{0\} = \{p \in \mathbb{R}^{2n} | \, y_i(p) = 0 \text{ for } i = 1, \cdots n\}$$

is Lagrangian. In fact, this Lagrangian submanifold is just one example from a particular category of Lagrangian submanifolds in $(\mathbb{R}^{2n}, \omega_0)$. Recall from Example 1.10 that, given a symmetric $n \times n$ matrix A, the set $\{(u, Au) | u \in \mathbb{R}^n\}$ is a Lagrangian subspace of the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$. This suggests a way of constructing Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_0)$. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and let ∇f be the corresponding gradient with respect to the standard metric on $\mathbb{R}^{2n}$. Then, the submanifold

$$\{(x, \nabla f(x)) | x \in \mathbb{R}^n\} \subset (\mathbb{R}^{2n}, \omega_0)$$

is Lagrangian. Indeed, for each $x \in \mathbb{R}^n$, the corresponding tangent space is

$$\{(u, H_f(x)u) | u \in \mathbb{R}^n\} \subset (\mathbb{R}^{2n}, \omega_0).$$

The matrix $H_f(x)$ is the Hessian matrix of f at x and, hence, is symmetric. One can see that the Lagrangian submanifold $\mathbb{R}^n \times \{0\}$ mentioned earlier simply corresponds to taking f to be any constant function.

Example 1.37. The real projective space $\mathbb{R}P^n$ naturally embeds in $\mathbb{C}P^n$:

$$\begin{array}{rcl} \mathbb{R}P^n & \to & \mathbb{C}P^n \\ [b_0 : \cdots : b_n] & \to & [b_0 : \cdots : b_n] \end{array}$$

where $b_i \in \mathbb{R}$ for $i = 0, \cdots n$. If we equip $\mathbb{C}P^n$ with the symplectic form ω defined in Example 1.32, then one can check that $\mathbb{R}P^n$ is a Lagrangian submanifold of $(\mathbb{R}P^n, \omega)$.

Example 1.38. Any closed Lagrangian submanifold of $(\mathbb{R}^{2n}, \omega_0)$ can be embedded as a Lagrangian submanifold of any symplectic manifold (M^{2n}, ω). This can be seen as follows.

By Darboux's theorem (Theorem 1.30), for any $x \in (M^{2n}, \omega)$, there is a neighborhood U which is symplectomorphic to an open set V of $(\mathbb{R}^{2n}, \omega_0)$. By translating and scaling, the manifold L can be embedded as a Lagrangian submanifold of $(\mathbb{R}^{2n}, \omega_0)$ such that $L \subset V$. Note that a scaling map is not a symplectomorphism but it preserves Lagrangian submanifolds. Indeed, if

$$\iota : L \to (\mathbb{R}^{2n}, \omega_0)$$

is the original Lagrangian embedding and

$$\begin{array}{rcl} f : \mathbb{R}^{2n} & \to & \mathbb{R}^{2n} \\ x & \to & \lambda x \end{array}$$

is a scaling map for some non-zero $\lambda \in \mathbb{R}$, we have

$$(f \circ \iota)^* \omega_0 = \iota^*(f^* \omega_0) = \lambda^2 \iota^* \omega_0 = 0$$

Therefore, the composition $f \circ \iota : L \to (\mathbb{R}^{2n}, \omega_0)$ is a Lagrangian embedding. If we compose this with the symplectomorphism $(V, \omega_0) \to (U, \omega)$, we get a Lagrangian embedding $j : L \to (M, \omega)$.

Example 1.39. Let M be a smooth manifold. Consider the cotangent bundle T^*M with the symplectic structure $\omega = -d\alpha$ of Example 1.31. If we identify M with the zero section of T^*M, then M is a Lagrangian submanifold of (T^*M, ω). More generally, a 1-form $\beta \in \Omega^1(M)$ defines an embedding

$$\begin{aligned} \beta : M &\to T^*M \\ x &\to (x, \beta(x)). \end{aligned}$$

This embedding is Lagrangian if

$$\begin{aligned} \beta^*\omega &= 0 \\ \Leftrightarrow \beta^*(-d\alpha) &= 0 \\ \Leftrightarrow d\beta^*(\alpha) &= 0. \end{aligned}$$

One can check that $\beta^*\alpha = \beta$. Therefore, the embedding is Lagrangian if and only if the 1-form β is closed.

In the case where $M = \mathbb{R}^n$ with linear coordinates $x_1, \cdots, x_n$, we get that (T^*M, ω) is just the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$. A 1-form of $\mathbb{R}^n$ can be identified with a smooth map $h : \mathbb{R}^n \to \mathbb{R}^n$ and the condition of being a closed 1-form corresponds to the condition that h is the gradient of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$. Hence, we just get the results of Example 1.36.

Suppose (M, ω) is a symplectic manifold and $L \subset M$ a Lagrangian submanifold. By Theorem 1.30, given any $p \in L$, there exists a neighborhood U of p in M which is symplectomorphic to a neighborhood V of p in T^*L. Here, T^*L is equipped with the standard symplectic form and L is identified with the zero section of T^*L just as in the previous example. The following theorem gives a stronger result.

Theorem 1.40 (Lagrangian embedding theorem). *Suppose (M, ω) is a symplectic manifold and $L \subset M$ a Lagrangian submanifold. Let ω_{T^*L} be the* *standard symplectic form on T^*L. Then there exist a neighborhood U of L in M, a neighborhood V of L in T^*L and a symplectomorphism*

$$\phi : (U, \omega) \to (V, \omega_{T^*L})$$

such that $\phi(x) = x$ for all $x \in L$.

1.4 Almost complex structures

In section 1.1, we defined the notion of a complex structure on a vector space. We also showed that, given any symplectic vector space (V, ω), there exists a complex structure J on V compatible with ω.(See Definition 1.11 and Lemma 1.13.) We will now give the corresponding definition and result for symplectic manifolds.

Definition 1.41. An almost complex structure J on a even-dimesional manifold M is a complex structure J_x on each tangent space T_xM which varies smoothly with $x \in M$. If (M, ω) is a symplectic manifold, an almost complex structure J is said to be compatible with ω if and only if J_x is compatible with ω_x for each $x \in M$.

Lemma 1.42. *Suppose (M, ω) is a symplectic manifold. Then there exists an almost complex structure J compatible with ω such that $g(.,.) = \omega(., J.)$.*

Given an almost complex structure J on a symplectic manifold (M, ω), we can determine the total Chern class $c(TM, J)$ of the complex vector bundle $(TM, J) \to M$. Based on the following lemma, the total Chern class $c(TM, J)$ does not depend on J.

Lemma 1.43. *The space of ω-compatible almost complex structures on a symplectic manifold is contractible.*

Hence, given any two ω-compatible almost complex structures J_0 and J_1, the corresponding complex vector bundles (TM, J_0) and (TM, J_1) are isomorphic. Indeed, by the above lemma, there is a continuous path of ω-compatible almost complex structures J_t on M joining J_0 to J_1. This one-paramater family gives the vector bundle $TM \times I \to M \times I$, a complex structure J defined by

$$\begin{aligned} J : TM \times I &\to TM \times I \\ (v, t) &\to J_{t_{\pi(v)}} v \end{aligned}$$

where $\pi : TM \to M$ is the projection map. Note that the restrictions of the above complex vector bundle to $M \times \{0\}$ and $M \times \{1\}$ are (TM, J_0) and (TM, J_1) respectively. Recall the following result from algebraic topology:

Lemma 1.44. *[10] Suppose M is a manifold and $E \to M \times I$ is a real or complex vector bundle where $I = [0,1]$. Then, the restrictions of the vector bundle over $M \times \{0\}$ and $M \times \{1\}$ are isomorphic.*

This implies that the complex vector bundles (TM, J_0) and (TM, J_1) are isomorphic. We conclude that the the total Chern class $c(TM, J)$ of the complex bundle $(TM, J) \to M$ is independent of the ω-compatible almost complex structure J.

Just as in Definition 1.18, we can define totally real immersions and totally real embeddings in a smooth even dimensional manifold M equipped with an almost complex structure J.

Definition 1.45. Suppose J is an almost complex structure on a manifold M. An immersion (embedding) $L \to M$ is totally real if for each $x \in L$, the vector space $df_x(T_xL)$ is a totally real subspace of $T_{f(x)}M$.

Note that Lagrangian immersions and embeddings in a symplectic manifold (M, ω) are totally real with respect to any ω-compatible almost complext structure.

1.5 Hamiltonian group actions and moment maps

We will now consider actions of Lie groups on symplectic manifolds that preserve the symplectic structure. More precisely, we will look at a particular class of these group actions, namely, the class of Hamiltonian group actions. Under certain conditions, Hamiltonian group actions can be used to construct new symplectic manifolds via an operation called symplectic reduction. Just as before, we will mainly state the basic definitions and results. The reader may consult [2] and [3] for the proofs. Also, basic notions of Lie group theory and group action on smooth manifolds can be found in [12].

Definition 1.46. Suppose G is a Lie group acting on a symplectic manifold (M, ω). The action is said to be symplectic if, for each $g \in G$, the corresponding diffeomorphism

$$\begin{array}{rcl} \psi_g : M & \to & M \\ x & \to & g \cdot x \end{array}$$

is a symplectomorphism.

If (M, ω) is a symplectic manifold with a G-group action, then any element X of the Lie algebra $\mathfrak{g}$ of G induces a vector field $\tilde{X}$ on M given by

$$\tilde{X}(x) = \frac{d}{dt}\psi_{exp(tX)} \cdot x|_{t=0} \quad \text{for } x \in M$$

where $exp : \mathfrak{g} \to G$ is the exponential map of the Lie group. If the group action is symplectic, we get a symplectic vector field since the corresponding flow $\{\psi_{exp(tX)}\}_{t\in\mathbb{R}}$ consists of symplectomorphisms. Therefore, to every vector $X \in \mathfrak{g}$, we can assign the closed 1-form $\iota_{\tilde{X}}\omega$.

Definition 1.47. The action of a Lie group G on a symplectic manifold (M, ω) is said to be Hamiltonian if there is a Lie algebra morphism

$$\begin{array}{rcl} H : \mathfrak{g} & \to & C^{\infty}(M) \\ X & \to & H_X \end{array}$$

such that $\iota_{\tilde{X}}\omega = dH_X$.

Here, the Lie algebra structure on the vector space $C^{\infty}(M)$ is defined by

$$\begin{array}{rcl} \{.,.\} : C^{\infty}(M) \times C^{\infty}(M) & \to & \mathbb{R} \\ (f, g) & \to & \omega(X_f, X_g) \end{array}$$

where X_f and X_g denote the Hamiltonian vector fields of f and g respectively.

We see that, given a Hamiltonian G-group action on (M, ω) and any vector X of $\mathfrak{g}$, the induced vector field $\tilde{X}$ is Hamitonian. In particular, the corresponding flow $\{\psi_{exp(tX)}\}_{t\in\mathbb{R}}$ consists of symplectomorphisms.

If the group G is connected, the Hamiltonian group action is a symplectic group action. This is due to the fact that any element $g \in G$ can be written as a product

$$g = exp(X_1) \cdots exp(X_n) \quad \text{where } X_i \in \mathfrak{g}$$

So, the corresponding diffeomorhism ψ_g can be written as

$$\psi_g = \psi_{exp(X_1)} \circ \cdots \circ \psi_{exp(X_n)}$$

We deduce that the map ψ_g is a symplectomorphism since it is a composition of symplectomorphisms. Hence, we can view a Hamiltonian group action of a connected Lie group G as as a symplectic G-group action where the linear map

$$\begin{array}{rcl} \mathfrak{g} & \to & \Omega^1(M) \\ X & \to & \iota_{\tilde{X}}\omega \end{array}$$

factors as

$$\begin{array}{ccll} \mathfrak{g} & \xrightarrow{H} & C^\infty(M) & f \\ & \searrow & \downarrow d & \downarrow \\ & & \Omega^1(M) & df \end{array}$$

Also, note that any element x of the manifold M defines a linear map

$$\begin{array}{rcl} \mathfrak{g} & \to & \mathbb{R} \\ X & \to & H_X(x) \end{array}$$

In other words, we can assign to every $x \in M$ an element of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

Definition 1.48. Let (M, ω) be a symplectic manifold with a Hamiltonian G-group action and corresponding Lie algebra morphism $H : \mathfrak{g} \to C^\infty(M)$. Let

$$\begin{array}{rcl} <.,.>: \mathfrak{g}^* \times \mathfrak{g} & \to & \mathbb{R} \\ (\eta, X) & \to & \eta(X) \end{array}$$

be the natural pairing for the Lie alebra $\mathfrak{g}$. Then, the smooth map $\mu : M \to \mathfrak{g}^*$ defined by

$$< \mu(x), X >= H_X(x)$$

for all $x \in M$ and $X \in \mathfrak{g}$ is called a moment map of the Hamiltonian group action.

It can be shown that the map $\mu : M \to \mathfrak{g}^*$ is G-equivariant with respect to the Hamiltonian group action of G on (M, ω) and the coadjoint action of G on $\mathfrak{g}^*$. That is

$$\mu(g \cdot x) = Ad_g^*(\mu(x))$$

for all $g \in G$ and $x \in M$. Here, Ad_g^* is the coadjoint map defined by g. This results from requiring that the map $H : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra morphism.

Example 1.49. Suppose f is a smooth function on a symplectic manifold with Hamiltonian vector field X_f. The corresponding flow of symplectomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$ defines a group action of $\mathbb{R}$ on (M, ω). This action is Hamiltonian with moment map $f : M \to \mathbb{R}$.

Example 1.50. The group $\mathbb{S}^1 = \{w \in \mathbb{C} \mid |w| = 1\}$ acts on $(\mathbb{C}^{n+1}, \omega_0)$ via

$$w \cdot (z_0, \cdots, z_n) = (wz_0, \cdots, wz_n)$$

This action is Hamiltonian with moment map

$$\mu(z_0, \cdots, z_n) = \frac{1}{2}\sum_{k=0}^{n} |z_k|^2$$

Suppose ξ is a regular value of the moment map μ of a Hamiltonian G-group action on (M, ω). If $\mu^{-1}(\xi)$ is non-empty, it is a smooth submanifold of M. Let $G_\xi \subset G$ be the stabilizer of ξ with respect to the coadjoint action. Then, the equivariance of μ implies that

$$\begin{aligned} \mu(g \cdot x) &= Ad_g^*(\mu(x)) \\ &= Ad_g^*(\xi) \\ &= \xi \end{aligned}$$

for all $g \in G_\xi$ and $x \in \mu^{-1}(\xi)$. Thus, the action of G_ξ preserves $\mu^{-1}(\xi)$. We will now state the symplectic reduction theorem.

Theorem 1.51. *[3] Suppose there is a Hamiltonian G-group action on (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$. Suppose ξ is a regular value of the moment map μ such that its stabilizer G_ξ acts freely on $\mu^{-1}(\xi)$. Then the smooth manifold $\mu^{-1}(\xi)/G_\xi$ has a natural symplectic structure σ defined by $\pi^*\sigma = \iota^*\omega$ where $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G_\xi$ is the projection map and $\iota : \mu^{-1}(\xi) \to M$ is the inclusion map.*

Example 1.52. Consider Example 1.50. The dual Lie algebra of $\mathbb{S}^1$ is isomorphic to $\mathbb{R}$. For any $r \neq 0$, the number $\frac{r^2}{2}$ is a regular value of μ and $\mu^{-1}(\frac{1}{2}r^2)$ is $\mathbb{S}_r^{2n+1}$, the $(2n+1)$-sphere centered at the origin with radius r. $\mathbb{S}^1$ is an abelian group. So the isotropy subgroup of $\frac{r^2}{2}$ is the whole group. The space $\mathbb{S}_r^{2n+1}/\mathbb{S}^1$ is the complex projective space $\mathbb{C}P^n$. The symplectic structure is $r^2\omega$ where ω is the form defined in Example 1.32.

Given a Hamiltonian G-group action on a symplectic manifold (M, ω), we can construct Lagrangian submanifolds of (M, ω) as follows. Let ξ be a regular value of the moment map μ with stabilizer $G_\xi = G$. Suppose we can construct the symplectic manifold $(\mu^{-1}(\xi)/G, \sigma)$. Given a Lagrangian submanifold L of $(\mu^{-1}(\xi)/G, \sigma)$, consider the preimage $\pi^{-1}(L) \subset \mu^{-1}(\xi)$. This set is a smooth submanifold of $\mu^{-1}(\xi)$. Let $j : \pi^{-1}(L) \to M$ be the inclusion map. Then we have $j^*\omega = 0$. Indeed, we have the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(L) & \stackrel{\tilde{\nu}}{\to} & \mu^{-1}(\xi) \\ \pi \downarrow & & \downarrow \pi \\ L & \stackrel{\nu}{\to} & \mu^{-1}(\xi)/G \end{array}$$

where ν and $\tilde{\nu}$ are inclusion maps. This implies that

$$\begin{aligned} j^*\omega &= (\iota \circ \tilde{\nu})^*\omega \\ \Rightarrow j^*\omega &= \tilde{\nu}^*(\iota^*\omega) \\ \Rightarrow j^*\omega &= \tilde{\nu}^*(\pi^*\sigma) \end{aligned}$$

Based on the definition of the symplectic form σ, we can rewrite the previous equation as

$$\begin{aligned} j^*\omega &= (\pi \circ \tilde{\nu})^*(\sigma) \\ \Rightarrow j^*\omega &= \nu^*(\sigma) \\ \Rightarrow j^*\omega &= 0 \end{aligned}$$

The last equation simply follows from the fact that L is a Lagrangian submanifold of $(\mu^{-1}(\xi)/G, \sigma)$. So we do get $j^*\omega = 0$. Furthermore, the dimension of $\pi^{-1}(L)$ is given by

$$\dim \pi^{-1}(L) = \dim L + \dim G$$

since $\pi^{-1}(L)$ is a principal G-bundle over L. Note that

$$\begin{aligned} \dim L &= \frac{\dim \mu^{-1}(\xi)/G}{2} \\ \Rightarrow \dim L &= \frac{\dim M - 2\dim G}{2} \\ \Rightarrow \dim L &= \frac{\dim M}{2} - \dim G \end{aligned}$$

Hence, the manifold $\pi^{-1}(L)$ has dimension $\frac{\dim M}{2}$. The set $\pi^{-1}(L)$ is a Lagrangian submanifold of (M, ω).

Example 1.53. Consider Example 1.52 with $r = 1$ for simplicity. As mentioned in Example 1.37, $\mathbb{R}P^n$ is a Lagrangian submanifold of $(\mathbb{C}P^n, \omega)$. Hence, $\pi^{-1}(\mathbb{R}P^n)$ is a Lagrangian submanifold of $(\mathbb{C}^{n+1}, \omega_0)$. The manifold $\pi^{-1}(\mathbb{R}P^n)$ consists of the set of points $(z_0, \cdots, z_n)$ of $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ such that there exists $\lambda \in \mathbb{S}^1$ with $\lambda \cdot (z_0, \cdots, z_n) = (x_0, \cdots, x_n)$ where $x_i \in \mathbb{R}$. From this we see that $\pi^{-1}(\mathbb{R}P^n)$ is isomorphic to $(\mathbb{S}^1 \times \mathbb{S}^n)/\sim$ where $(z, x) \sim (-z, -x)$. The Lagrangian embedding is given by

$$\begin{aligned} (\mathbb{S}^1 \times \mathbb{S}^n)/\sim \to &\quad \mathbb{C}^{n+1} \\ \{z, x\} \to &\quad zx. \end{aligned}$$

2 Lagrangian immersions

One of the basic problems in symplectic geometry is the following: Given a sympletic manifold (M, ω), what manifolds can be embedded as Lagrangian submanifolds in (M, ω)? Before addressing this problem, we will first consider the simpler case of Lagrangian immersions. We start out this section by giving some topological results for Lagrangian immersions that are simply due to the fact that a Lagrangian subbundle of a symplectic vector bundle is totally real with respect to any compatibe complex structure ([11]). We then state Gromov's h-principle which gives a homotopy classification of the space of Lagrangian immersions. We will not give a proof of the h-principle. (We refer the reader to Gromov's paper [7] and Eliashberg's book [6].) Instead, we list some simple consequences of the h-principle for Lagrangian immersions in $(\mathbb{C}^n, \omega_0)$ ([2]).

Suppose $f : L \to (M, \omega)$ is a Lagrangian immersion. Pulling back via the map f, we get the symplectic vector bundle $(f^*TM, \omega) \to L$ over L. Furthermore, the bundle map

$$\begin{array}{ccc} TL & \xrightarrow{df} & f^*TM \\ \downarrow & & \downarrow \\ L & = & L \end{array}$$

embeds TL as a Lagrangian subbundle of f^*TM. The following result is a direct consequence of Corollary 1.21.

Lemma 2.1. *Let (M^{2n}, ω) be a symplectic manifold and J an almost complex structure compatible with ω. If $f : L^n \to (M, \omega)$ is a Lagrangian immersion, then the pullback bundle $(f^*TM, J) \to L$ is isomorphic to the complexification of the tangent bundle of L.*

Note that Lemma 2.1 implies that $c(f^*TM) = c(TL \otimes_{\mathbb{R}} \mathbb{C})$. These are the total Chern classes of the complex vector bundles $TL \otimes_{\mathbb{R}} \mathbb{C}$ and (f^*TM, J) respectively. In particular, since $c(f^*TM) = f^*c(TM)$ and the odd-dimensional Chern classes of a complexified vector bundle are zero, we get that $f^*c_i(TM) = 0$ if i is odd. The following is also a consequence of Lemma 2.1.

Corollary 2.2. *Suppose L is an oriented manifold and $f : L \to (M, \omega)$ a Lagrangian immersion. Then $\nu(f)$ is isomorphic to $(-1)^{\frac{n(n-1)}{2}} TL$ as oriented vetor bundles.*

Here, $\nu(f) \to L$ is the normal bundle of the immersion f. It is oriented as follows. First, we equip the vector bundle TM with the orientation induced by J: if $(e_1, \cdots, e_n)$ is a basis of the complex fiber (T_xM, J_x), for $x \in M$, then $(e_1, J_xe_1, \cdots, e_n, J_xe_n)$ is an oriented basis of the real fiber T_xM. The ordered list of vectors $(w_1, \cdots, w_n)$ is an oriented basis of $\nu(f)_x$ for $x \in L$ if and only if, given any oriented basis $(v_1, \cdots, v_n)$ of T_xL, $(d_xf(v_1), \cdots, d_xf(v_n), w_1, \cdots, w_n)$ is an oriented basis of $(f^*TM)_x$.

Proof. Consider the real vector bundle map

$$\begin{aligned} \Phi : TL &\to f^*TM \\ (x, v) &\to J_{f(x)} df_x(v) \end{aligned}$$

Hence, for each $x \in L$, Φ_x maps T_xL to $J_{f(x)} \circ df_x(T_xL)$. Since $df_x(T_xL)$ is a Lagrangian subspace of $T_f(x)M$ for every $x \in L$, we see that $J_{f(x)} \circ df_x(T_xL)$ is perpendicular to $df_x(T_xL)$. Here, the orthogonality is with respect to the real inner product induced by ω and J. Hence, the map Φ gives an isomorphismm $TL \simeq \nu(f)$ as unoriented vector bundles. Now, if we take into account the orientations of these bundles, we see that an oriented basis $(v_1, \cdots, v_n)$ of T_xL gives the oriented basis

$$(d_xf(v_1), J_{f(x)}d_xf(v_1), \cdots, d_xf(v_n), J_{f(x)}d_xf(v_n))$$

of $(f^*TM))_x$. After $\frac{n(n-1)}{2}$ transpositions, we get the basis

$$(d_xf(v_1), \cdots, d_xf(v_n), J_{f(x)}d_xf(v_1), \cdots, J_{f(x)}d_xf(v_n)).$$

Based on the orientation convention for $\nu(f)$, we deduce that the ordered list $((-1)^{\frac{n(n-1)}{2}} J_{f(x)} d_x f(v_1), \cdots, J_{f(x)} d_x f(v_n))$ gives an oriented basis of $\nu(f)_x$ for every $x \in L$. We conclude that the bundle map Φ gives an isomorphism $(-1)^{\frac{n(n-1)}{2}} TL \simeq \nu(f)$ as oriented vector bundles. □

Consider the case where the symplectic manifold is the space $\mathbb{C}^n$ equipped with the standard symplectic structure ω_0. For any map $f : L \to \mathbb{C}^n$, the pullback symplectic vector bundle $(f^*T\mathbb{C}^n, \omega_0) = L \times (\mathbb{C}^n, \omega_0) \to L$ is trivial. Hence, if a manifold admits a Lagrangian immersion in $(\mathbb{C}^n, \omega_0)$, its complexified tangent bundle must be trivial by Lemma 2.1. Gromov's h-principle implies that the converse is true.

Theorem 2.3. *[7] A closed manifold L^n admits a Lagrangian immersion in $(\mathbb{C}^n, \omega_0)$ if and only if its complexified tangent bundle is trivial.*

Before stating the h-principle of Gromov and giving a proof of Theorem 2.3, we will list some implications of this theorem.

Corollary 2.4. *For any stably parallelisable closed manifold L of dimension n, there exists a Lagrangian immersion $L \to (\mathbb{C}^n, \omega_0)$.*

Recall that a manifold L is stably parallelisable if the tangent bundle is stably trivial: For some nonnegative integer k, the bundle $TL \otimes (L \times \mathbb{R}^k) \to L$ is trivial.

Proof. Suppose L^n is a stably parallelisable manifold. A real vector bundle isomorphism

$$\begin{tikzcd} TL \oplus (L \times \mathbb{R}^k) \arrow[r] \arrow[dr] & L \times \mathbb{R}^{n+k} \arrow[d] \\ & L \end{tikzcd}$$

induces a complex vector bundle isomorphism

$$\begin{tikzcd} (TL \otimes_{\mathbb{R}} \mathbb{C}) \oplus (L \times \mathbb{C}^k) \arrow[r] \arrow[dr] & L \times \mathbb{C}^{n+k} \arrow[d] \\ & L \end{tikzcd}$$

If a complex vector bundle $E \to L$ of rank $r \geq \frac{\dim L}{2}$ is stably trivial, then it is trivial ([9]). Hence, $TL \otimes_{\mathbb{R}} \mathbb{C} \to L$ is a trivial complex vector bundle. We get our result by applying Theorem 2.3. □

Example 2.5.

1. Any n-dimensional Lie group G admits a Lagrangian immersion in $(\mathbb{C}^n, \omega_0)$ since the tangent bundle of a Lie group is trivial: Let $\mathfrak{g} = T_e G$ be the Lie algebra of G. Then the bundle map

$$\begin{array}{rcl} TG & \to & G \times \mathfrak{g} \\ (g, v) & \to & (g, d_g L_{g^{-1}}(v)) \end{array}$$

is an isomorphism. Here, $L_{g^{-1}} : G \to G$ is the left translation map by g^{-1}.

2. The unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is stably parallelisable for every $n \in \mathbb{N}$. Indeed, the bundle map

$$\begin{array}{rcl} \mathbb{S}^n \times \mathbb{R}^{n+1} & \to & T\mathbb{S}^n \oplus \mathbb{R}^n \\ (x, v) & \to & (x, v - < x, v > x) \end{array}$$

is an isomorphism. As in the previous sections, $< .,. >$ denotes the standard inner product on $\mathbb{R}^{n+1}$. Corollary 2.4 implies that there is a Lagrangian immersion of $\mathbb{S}^n$ into $(\mathbb{C}^n, \omega_0)$. For instance, if we view the n-sphere as the set

$$\mathbb{S}^n = \{(\vec{x}, y) \in \mathbb{R}^n \times \mathbb{R} \mid \| \vec{x} \|^2 + y^2 = 1\}$$

then the map

$$\begin{array}{rcl} \mathbb{S}^n & \to & \mathbb{C}^n \\ (\vec{x}, y) & \to & \vec{z} = (1 + iy) \cdot \vec{x} = (x_1 + ix_1 y, \cdots, x_n + ix_n y) \end{array}$$

is a Lagrangian immersion. This map is called the Whitney immersion.

3. Any orientable surface $\Sigma \in \mathbb{R}^3$ is stably parallelisable. Just as in the previous case, we can use a normal vector field to construct a bundle iomorphism $T\Sigma \oplus \mathbb{R} \to \Sigma \times \mathbb{R}^3$. Thus, Σ admits a Lagrangian immersion in $(\mathbb{C}^4, \omega_0)$.

We will now introduce the h-principle of Gromov. We start out by recalling the notion of a 1-jet bundle and of a differential relation on a 1-jet bundle. The following definitions are taken from [6] and [18].

Definition 2.6. Suppose $E^{n+k} \xrightarrow{\pi} M^n$ is a smooth fiber bundle. For any $p \in M$, denote by $\Gamma_p(\pi)$ the set of sections of the fiber bundle which are defined on a neighborhood of p. Then two local sections $\phi, \psi \in \Gamma_p(\pi)$ are said to be 1-equivalent at p if $\phi(p) = \psi(p)$ and $d_p\phi = d_p\psi$. The equivalence class containing $\phi \in \Gamma_p(\pi)$ is called the 1-jet of ϕ at p and is denoted by $j^1_p\phi$.

Suppose V is a neighborhood of $p \in M$ and

$$\begin{array}{rcl} (x^i, u^\alpha) : \pi^{-1}(U) & \to & \mathbb{R}^n \times \mathbb{R}^k \\ \downarrow & & \downarrow \\ x^i : U & \to & \mathbb{R}^n \end{array}$$

is a local coordinate system for the bundle. It can be shown that, if the local sections $\phi, \psi \in \Gamma_p(\pi)$ satisfy $\phi(p) = \psi(p)$, then the condition $d_p\phi = d_p\psi$ is equivalent to

$$\frac{\partial(u^\alpha \circ \phi)}{\partial x^i}|_p = \frac{\partial(u^\alpha \circ \psi)}{\partial x^i}|_p$$

for $1 \leq i \leq n$ and $1 \leq \alpha \leq k$. Using this, one can prove the following.

Lemma 2.7. *Suppose $E^{n+k} \xrightarrow{\pi} M^n$ is a smooth fiber bundle. Then the set*

$$J^1E = \{j^1_p\phi \,|\, p \in M,\ \phi \in \Gamma_p(\pi)\}$$

is a smooth manifold of dimension $n + k + nk$ and is called the first jet manifold of the bundle. In fact, the map

$$\begin{array}{rcl} \pi_1 : J^1E & \to & M \\ j^1_p\phi & \to & p \end{array}$$

gives J^1E the stucture of a smooth fiber bundle over M. Furthermore, there is a well-defined projection map

$$\begin{array}{rcl} \pi_{1,0} : J^1E & \to & E \\ j^1_p\phi & \to & \phi(p). \end{array}$$

Definition 2.8. A section $f : M \to E$ of the bundle $E^{n+k} \xrightarrow{\pi} M^n$ induces a section

$$\begin{array}{rcl} j^1f : M & \to & J^1E \\ p & \to & j^1_pf \end{array}$$

of the bundle $J^1E \xrightarrow{\pi_1} M$. This section is called the 1-jet extension of f. Conversely, a section $F : M \to J^1E$ induces the section

$$bsF = \pi_{1,0} \circ F : M \to E.$$

The section F is said to be holonomic if $F = j^1(bsF)$.

Example 2.9. Let L and M be smooth manifolds. Consider the trivial fibration $E = L \times M \to M$. A section of this bundle simply corresponds to a map $f : L \to M$. Consequently, the fiber $(J^1E)_p$ over a point $p \in L$ is a triple (p, q, A) where $q \in M$ and $A : T_pL \to T_qM$ is a linear map. Furthermore, a section of $J^1E \to M$ can be identified with a bundle map

$$\begin{array}{ccc} TL & \xrightarrow{\Phi} & TM \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & M \end{array}$$

This section is holonomic if $\Phi = d\phi$.

Definition 2.10. A differential relation of order 1 for a a smooth fiber bundle $E \to M$ is a subset $\mathcal{R}$ of the first jet manifold J^1E. A formal solution of the differential relation $\mathcal{R}$ is a section $F : M \to \mathcal{R} \subset J^1E$. The solution is said to be genuine if F is holonomic. The space of formal solutions for a differential relation $\mathcal{R}$ will be denoted by $Sec\mathcal{R}$.

For any $r \in \mathbb{N}$, we can define the r-jet space of a fiber bundle and the notion of a differential relation of order r. But, for our purpose, we only need to consider the case where $r = 1$.

Example 2.11. Consider the trivial fibration $E = L \times M \to M$ from Example 2.9.

1. If $\dim L \leq \dim M$, we can define the immersion relation $\mathcal{R}_{imm}$. A formal solution can be identified with a fiberwise injective bundle map

$$\begin{array}{ccc} TL & \xrightarrow{\Phi} & TM \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & M \end{array} \tag{2.1}$$

   Furthermore, one can see that genuine solutions are simply immersions $L \to M$.

2. Suppose (M^{2n}, ω) is a symplectic manifold and $\dim L = n$. We can define the relation $\mathcal{R}_{lag} \subset \mathcal{R}_{imm}$. A formal solution of $\mathcal{R}_{lag}$ can be identified with a fiberwise injective bundle map just as in 2.1 but with the additional requirement that the bundle map Φ maps each tangent space T_pL to a Lagrangian subspace of $T_{\phi(p)}M$. A genuine solution of the differential relation is simply a Lagrangian immersion $L \to (M, \omega)$.

Based on the previous example, we see that the problem of existence of Lagrangian immersions $L \to (M^{2n}, \omega)$ can be reformulated as a problem of existence of genuine solutions of the differential relation $\mathcal{R}_{lag}$. For any fiber bundle $E \to M$ and a differential relation $\mathcal{R}$, the existence of a formal solution is a necessary condition for the existence of a genuine solution. The problem of existence of a formal solution is a simpler and homotopy-theoretical problem. Furthermore, it turns out that, for many differential relations, the existence of a formal solution is a sufficient condition for the existence of a genuine solution.

Definition 2.12. [6] A differential relation $\mathcal{R}$ is said to satisfy the h-principle if every formal solution of $\mathcal{R}$ is homotopic in $Sec\mathcal{R}$ to a genuine solution of $\mathcal{R}$.

Theorem 2.13. *[6],[7] Suppose (M^{2n}, ω) is a symplectic manifold where ω is an exact form. Then, for any n-dimensional manifold L, the corresponding differential relation $\mathcal{R}_{lag}$ satisfies the h-principle.*

Proof of Theorem 2.3. The above theorem implies Theorem 2.3. Indeed, the symplectic form ω_0 on $\mathbb{C}^n$ is exact. Furthermore, since the symplectic vector bundle $(T\mathbb{C}^n, \omega_0) \to \mathbb{C}^n$ is trivial and any map $L \to \mathbb{C}^n$ is homotopically trivial, we see that any formal solution for the corresponding relation $\mathcal{R}_{lag}$ can be homotoped in $Sec\mathcal{R}_{lag}$ to a formal solution of the form

$$\begin{array}{ccc} TL & \to & \{*\} \times (\mathbb{C}^n, \omega_0) \\ \downarrow & & \downarrow \\ L & \to & * \end{array}$$

where $*$ is some point in $\mathbb{C}^n$. Hence, we see that the problem of existence of Lagrangian immersions $L \to (\mathbb{C}^n, \omega_0)$ reduces to the problem of existence of Lagrangian bundle maps

$$\begin{array}{ccc} TL & \to & L \times (\mathbb{C}^n, \omega_0) \\ & \searrow \quad \swarrow & \\ & L & . \end{array} \tag{2.2}$$

It can be shown that the existence of a totally real bundle map

$$\begin{array}{ccc} TL & \rightarrow & L \times (\mathbb{C}^n, i) \\ \searrow & & \swarrow \\ & L & \end{array} \tag{2.3}$$

implies the existence of a Lagrangian bundle map such as 2.2. This is essentially due to the fact that the space of Lagrangian subspaces in $(\mathbb{C}^n, \omega_0)$ is a deformation retract of the space of totally real subspaces of $\mathbb{C}^n$. Finally, we apply Lemma 1.20. A totally real bundle map such as 2.3 exists if and only there exists a complex vector bundle isomorphism

$$\begin{array}{ccc} TL \otimes_{\mathbb{R}} \mathbb{C} & \rightarrow & L \times \mathbb{C}^n \\ \searrow & & \swarrow \\ & L & . \end{array}$$

This completes the proof. □

For a general symplectic manifold (M, ω), the following version of the h-principle holds.

Theorem 2.14. *[6], [7] Let (M^{2n}, ω) be a symplectic manifold and L^n a closed manifold. Then the corresponding differential relation $\mathcal{R}_{lag}$ satisfies the h-principle if the condition $[\phi^*\omega] = 0 \in H^2(L; \mathbb{R})$ is imposed on the set of formal solutions*

$$\begin{array}{ccc} TL & \xrightarrow{\Phi} & (TM, \omega) \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & M \end{array}$$

of $\mathcal{R}_{lag}$.

We end this section with a result which basically states that Lagrangian immersions in $(\mathbb{C}^n, \omega_0)$ can be used to construct Lagrangian embeddings in $(\mathbb{C}^{n+r}, \omega_0)$ for any $r \geq 1$. We refer the reader to [2] for the proof.

Definition 2.15. Two Lagrangian immersions $f_0, f_1 : L \rightarrow (M, \omega)$ are said to be l-regularly homotopic if there exists a smooth homotopy

$$F : L \times [0, 1] \rightarrow (M, \omega)$$

such that $f_t = F(\cdot, t)$ is a Lagrangian immersion for each each $t \in [0, 1]$.

Lemma 2.16. *Suppose $f : L \to (\mathbb{C}^n, \omega_0)$ is a Lagrangian immersion and $g : W \to (\mathbb{C}^r, \omega_0)$ is a Lagrangian embedding where L and W are both closed manifolds. Then there is a Lagrangian embedding $L \times W \to (\mathbb{C}^{n+r}, \omega_0)$ which is l-regularly homotopic to $f \times g$.*

3 Lagrangian embeddings

In this section, we consider the Lagrangian embedding problem. We start out by listing some simple topological properties of Lagrangian submanifolds which follow directly from the previous results. Then, we outline a proof of Kawashima's theorem which states that there is no Lagrangian embedding $\mathbb{S}^n \to (\mathbb{C}^n, \omega)$ if $n \neq 1, 3$ and ω is any symplectic form on $\mathbb{C}^n$. These results are from [2] and [11].

Lemma 3.1. *Let (M, ω) be a closed symplectic manifold and $\iota : L \to M$ a Lagrangian embedding of an oriented closed manifold L. Consider the homology class $\alpha = \iota_*([L])$ where $[L]$ is the fundamental class of L. The self-intersection number of the Lagrangian L is given by*

$$\alpha \cdot \alpha = (-1)^{\frac{n(n-1)}{2}} \chi(L)$$

where $\chi(L)$ is the Euler characteristic of L

Proof. The self-intersection number of L is $\alpha \cdot \alpha = < PD(\alpha), \alpha >$ where

$$PD : H_n(M : \mathbb{Z}) \to H^n(M : \mathbb{Z})$$

is the Poincare duality isomorphism. Recall that $\iota^*(PD(\alpha)) = e(\nu(L))$ is the Euler class of the normal bundle of L in M. From Corollary 2.2, we know that

$$\nu(L) \simeq (-1)^{\frac{n(n-1)}{2}} TL$$

as oriented vector bundles. Hence,

$$\iota^*(PD(\alpha)) = e(\nu(L)) = (-1)^{\frac{n(n-1)}{2}} e(TL)$$

and we get

$$\begin{aligned} \alpha \cdot \alpha &= < PD(\alpha), \iota_*[L] > \\ \Rightarrow \alpha \cdot \alpha &= < \iota^*(PD(\alpha)), [L] > \\ \Rightarrow \alpha \cdot \alpha &= (-1)^{\frac{n(n-1)}{2}} < e(TL), [L] > \\ \Rightarrow \alpha \cdot \alpha &= (-1)^{\frac{n(n-1)}{2}} \chi(L). \end{aligned}$$

This completes the proof. $\square$

Corollary 3.2. *If L is an orientable smooth n-manifold which admits a Lagrangian embdedding into $(\mathbb{C}^n, \omega_0)$, then*

$$\chi(L) = 0.$$

In particular, this implies that the only closed orientable surface that admits a Lagrangian embedding in $(\mathbb{C}^4, \omega_0)$ is the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Proof. In Example 1.38, we showed that any Lagrangian submanifold L in $(\mathbb{C}^n, \omega_0)$ can be embedded as a Lagrangian submanifold of any closed symplectic manifold (M, ω). Furthermore, the self-intersection number of this induced Lagrangian submanifold in (M, ω) is zero since the image of the fundamental class $[L]$ in $H_n(M)$ is zero. Hence, we get the result by applying Lemma 3.1. $\square$

In the last section, we showed that $\mathbb{S}^n$ admits a Lagrangian immersion in $(\mathbb{C}^n, \omega_0)$ for any $n \in \mathbb{N}$. This is not the case for Lagrangian embeddings.

Theorem 3.3. *If $n \neq 1$, then $\mathbb{S}^n$ cannot be embedded in $(\mathbb{C}^n, \omega_0)$ as a Lagrangian submanifold.*

This result follows from an important theorem of Gromov which was obtained by using the theory of pseudo-holomorphic curves. (Consult e.g. [2] for the definition of a pseudo-holomorphic curve.) Before stating this theorem, we need to define the symplectic area class of a Lagrangian immersion in $(\mathbb{C}^n, \omega_0)$.

The standard symplectic form $\omega_0 = \sum dx^i \wedge dy^i$ can be written as $\omega = d\lambda$ where $\lambda = \sum x^i dy^i$. Suppose $f : L \to (\mathbb{C}^n, \omega_0)$ is a Lagrangian immersion. Then the 1-form $f^*\lambda \in \Omega^1(L; \mathbb{R})$ is closed since $df^*\lambda = f^*d\lambda = f^*\omega = 0$.

Definition 3.4. Let λ be be the 1-form $\sum x^i dy^i$ on $\mathbb{C}^n$. Given a Lagrangian immersion $f : L \to (\mathbb{C}^n, \omega_0)$, the cohomology class $\omega_0(f) = [f^*\lambda] \in H^1(L; \mathbb{R})$ is called the symplectic area class of f. The Lagrangian immersion is said to be exact if $\omega_0(f) = 0 \in H^1(L; \mathbb{R})$.

If $f : L \to (\mathbb{C}^n, \omega_0)$ is an exact Lagrangian embedding, we will simply say that L is an exact Lagrangian submanifold of $(\mathbb{C}^n, \omega_0)$.

Theorem 3.5. *[8] There are no closed exact Lagrangian submanifolds of $(\mathbb{C}^n, \omega_0)$.*

Theorem 3.3 follows from Theorem 3.5 since $H^1(\mathbb{S}^n, \mathbb{R}) = 0$ if $n \neq 1$. In particular, a Lagrangian embedding $f : \mathbb{S}^n \to (\mathbb{C}^n, \omega_0)$ would imply that $\omega_0(f) = 0$, thus contradicting Theorem 3.5.

We will now give the outline of a proof of Theorem 3.3 for the case $n \neq 1, 3$ which only uses algebraic and differential topology and which applies to any symplectic structure on $\mathbb{C}^n$.

Theorem 3.6. *[11] For $n \neq 1, 3$, the sphere $\mathbb{S}^n$ does not admit a Lagrangian embedding in $(\mathbb{C}^n, \omega)$ where ω is any symplectic structure on $\mathbb{C}^n$.*

Outline of the proof: First note that, for $\omega = \omega_0$, we can already rule out the case where n is even by using Corollary 3.2: $\chi(\mathbb{S}^n)$ is 0 for n odd and is 2 for n even. We will now show that the existence of a Lagrangian embedding of the n-sphere in $(\mathbb{C}^n, \omega)$ implies that its tangent bundle is trivial. Recall that we are using the identification $\mathbb{C}^n = \mathbb{R}^{2n}$ given by

$$(z_1, \cdots, z_n) \to (x_1, \cdots, x_n, y_1, \cdots, y_n) \tag{3.1}$$

where $z_j = x_j + iy_j$. If we view the n-sphere as

$$\mathbb{S}^n = \{(x_1, \cdots, x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$$

the standard embedding in $\mathbb{C}^n$ is given by

$$\begin{array}{rcl} \iota : \mathbb{S}^n & \to & \mathbb{C}^n = \mathbb{R}^{2n} \\ (x_1, \cdots, x_{n+1}) & \to & (x_1, \cdots, x_{n+1}, 0, \cdots, 0) \end{array}$$

Suppose there is a Lagrangian embedding $f : \mathbb{S}^n \to \mathbb{C}^n$. Smale proved that any embedding of $\mathbb{S}^n$ in $\mathbb{C}^n$ is regularly homotopic to ι ([19]). That is, there exists a smooth map

$$F : \mathbb{S}^n \times I \to \mathbb{C}^n$$

such that $F_0 = \iota$, $F_0 = f$ and $F_t : \mathbb{S}^n \to \mathbb{C}^n$ is an immersion for every $t \in I$. This smooth homotoy induces a smooth bundle map

$$\begin{array}{ccc} T\mathbb{S}^n \times I & \xrightarrow{\Phi} & (\mathbb{S}^n \times I) \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ \mathbb{S}^n \times I & = & \mathbb{S}^n \times I \end{array}$$

given by

$$\begin{array}{ccc} ((x,u),t) & \to & (x,t,d_x f_t(u)) \\ \downarrow & & \downarrow \\ (x,t) & = & (x,t) \end{array}$$

where $u \in T_x\mathbb{S}^n$, $x \in \mathbb{S}^n$. Since the bundle map Φ is injective on each fiber, we can construct the corresponding normal bundle $E \to \mathbb{S}^n \times I$. We have $E|_{\mathbb{S}^n\times\{0\}} = \nu(\iota)$ and $E|_{\mathbb{S}^n\times\{1\}} = \nu(f)$. By Lemma 1.44, this implies that $\nu(\iota)$ is isomorphic to $\nu(f)$. Since f is a Lagrangian embedding, we get

$$\nu(\iota) \simeq \underbrace{\nu(f) \simeq T\mathbb{S}^n}_{\text{by Corollary 2.2}}.$$

The vector bundle $\nu(\iota)$ is trivial. Indeed, recall that, for $x \in \mathbb{S}^n$, the tangent space is given by

$$T_x\mathbb{S}^n = \{u \in \mathbb{R}^{n+1} | < x, u >= 0\}$$

where $< .,. >$ is the Euclidean inner product. Hence, a trivialization of the real bundle $\nu(\iota)$ is given by

$$\begin{array}{ccc} \mathbb{S}^n \times \mathbb{R}^n & \to & \nu(\iota) \subset \mathbb{S}^n \times \mathbb{R}^{2n} \\ \downarrow & & \downarrow \\ \mathbb{S}^n & = & \mathbb{S}^n \end{array}$$

$$(x, (a_1, \cdots, a_n)) \to (a_1x_1, \cdots, a_1x_{n+1}, a_2, \cdots, a_n)$$

where $x = (x_1, \cdots, x_{n+1}) \in \mathbb{S}^n$. Therefore, the existence of a Lagrangian embedding of $\mathbb{S}^n$ in $\mathbb{C}^n$ implies that the tangent bundle of $\mathbb{S}^n$ is trivial. Recall that the tangent bundle $T\mathbb{S}^n$ of the n-sphere is trivial if and only if $n = 1, 3$ or 7 ([10]). Hence, $\mathbb{S}^n$ does not admit a Lagrangian embedding if $n \neq 1, 3$ or 7. Note that this proof is based on the fact that a Lagrangian embedding is totally real with respect to an ω-compatible almost complex structure. In particular, in the case where we use the standard symplectic structure ω_0 and the standard complex structure i on $\mathbb{C}^n$, we get the following.

Theorem 3.7. *There is no totally real embedding $\mathbb{S}^n \to \mathbb{C}^n$ if $n \neq 1, 3$ or 7.*

To finish the proof of Theorem 3.6, we need to rule out the case $n = 7$. First, we need to define the Lagrangian Grassmannian.

Definition 3.8. The Lagrangian Grassmannian Λ_n is defined to be the set of all Lagrangian subspaces of $(\mathbb{C}^n, \omega_0)$.

In the case $n = 1$, the Lagrangian Grassmannian is simply the set of all lines in $\mathbb{R}^2$. So it is the set $\Lambda_1 = \mathbb{R}P^1 \simeq \mathbb{S}^1$. In general, the following holds.

Lemma 3.9. *The Lagrangian Grassmannian Λ_n can be identified with the homogeneous space $U(n)/O(n)$.*

This identification gives Λ_n the structure of a smooth manifold. Using this result, we will now finish outlining the proof of Theorem 3.6. For each $n \in \mathbb{N}$, we will denote by $G_{2n,n}$ the Grassmannian of all real n-dimensional subspaces of $\mathbb{R}^{2n} = \mathbb{C}^n$. Let $j_n : \Lambda_n \to G_{2n,n}$ be the natural inclusion and let $j_* : \pi_n(\Lambda_n) \to \pi_n(G_{2n,n})$ be the corresponding homomorphism on the n-th homotopy groups. Suppose ω is any symplectic structure on $\mathbb{C}^n$. The symplectic vector bundle $(T\mathbb{C}^n, \omega) \to \mathbb{C}^n$ is trivial. Indeed, recall that symplectic vector bundles $(E, \omega) \to M$ over a manifold M have stucture group $Sp(2n)$. Therefore, they are classified by homotopy classes of maps $M \to BSp(2n)$ where $BSp(2n)$ denotes the classifying space of the group $Sp(2n)$. Since any map $\mathbb{C}^n \to BSp(2n)$ is homotopically trivial, we deduce that any symplcetic bundle over $\mathbb{C}^n$ is trivial. Now, given an embedding $f : \mathbb{S}^n \to \mathbb{C}^n$, we can use a trivialisation of the symplectic vector bundle $(T\mathbb{C}^n, \omega) \to \mathbb{C}^n$ to define the Gauss map $\tilde{f} : \mathbb{S}^n \to G_{2n,n}$ which maps each $x \in \mathbb{S}^n$ to the corresponding tangent plane $df_x(T_x\mathbb{S}^n)$. If f is Lagrangian, the map f factors as

$$\mathbb{S}^n \to \Lambda_n \to G_{2n,n}$$

Hence $[\tilde{f}] \in Im(j_*)$. Note that $[\tilde{f}] = [\tilde{\iota}]$ where $\tilde{\iota}$ is the Gauss map for the standard embedding ι: a regular homotopy between f and ι induces a homotopy between $\tilde{f}$ and $\tilde{\iota}$.

In [11], the homomorphisms $j_* : \pi_n(\Lambda_n) \to \pi_n(G_{2n,n})$ are computed for the cases $n = 3, 7$. It is shown that:

1. The group $\pi_3(\Lambda_3)$ is isomorphic to $\mathbb{Z}_4$. The groups $\pi_7(\Lambda_7)$, $\pi_3(G_{6,3})$ and $\pi_7(G_{14,7})$ are all isomorphic to $\mathbb{Z}_2$.
2. $j_* : \pi_3(\Lambda_3) \to \pi_3(G_{6,3})$ is surjective.
   $j_* : \pi_7(\Lambda_7) \to \pi_7(G_{14,7})$ is the zero map.
3. $[\tilde{\iota}]$ is the generator of $\pi_n(G_{2n,n})$ for $n = 3, 7$

We now see that there cannot be a Lagrangian embedding $f : \mathbb{S}^7 \to (\mathbb{C}^7, \omega)$. Indeed, on one hand, this would imply that $[\tilde{f}] = [\tilde{\iota}]$ is the generator of $\pi_7(G_{14,7})$. In particular, $[\tilde{f}]$ would be non-zero. But, on the other hand, the fact that $[\tilde{f}] \in Im(j_*)$ and $j_* : \pi_7(\Lambda_7) \to \pi_7(G_{14,7})$ is the zero map would imply that $[\tilde{f}] = 0$. So we get a contradiction and this completes the proof.□

Note that we could not rule out the case $n = 3$. In fact, Mueller showed that there exists a symplectic structure ω on $\mathbb{C}^3$ for which $\mathbb{S}^3$ admits a Lagrangian embedding in $(\mathbb{C}^3, \omega)$ ([14]).

We finish this section by giving a proof of Lemma 3.9.
Proof of Lemma 3.9: Recall that the standard Hermitian inner product $(.,.)$ can be written as

$$(.,.) = <.,.> - i\omega_0(.,.)$$

where $<.,.>$ is the standard Euclidean product on $\mathbb{R}^{2n}$. From this relation, we see that an orthonormal basis $\{e_1, \cdots, e_n\}$ of a Lagrangian subspace L determines a unitary basis for the complex vector space $\mathbb{C}^n$. Conversely, the real vector space spanned by a unitary basis determines a Lagrangian subspace of $\mathbb{C}^n$. This implies that the standard action of the unitary group $U(n)$ on $\mathbb{C}^n$ induces a transitive action on Λ_n. The stabilizer of the Lagrangian subspace $\mathbb{R}^n \times \{0\}$ under this group action is $O(n)$. Hence, $\Lambda_n \simeq U(n)/O(n)$.□

4 Monotone symplectic manifolds and monotone Lagrangian submanifolds

In the last two sections, we listed some topological properties of Lagrangian immersions and embeddings. Most of these results are obtained by using tools in differential and algebraic topology and are simply due to the fact that a Lagrangian immersion (embedding) $L \to (M, \omega)$ is totally real with respect to some ω-compatible almost complex structure J. Symplectic topologists have to rely on more complex tools such as pseudo-holomorphic curves analysis and Floer (co)homology so they can obtain other topological restrictions on Lagrangian submanifolds. In order to apply these advanced methods, certain conditions are imposed on the symplectic manifold or Lagrangian submanifold. For instance, these manifolds are often required to be monotone. In this section, we will define this concept as well as the concept of the Maslov number of monotone Lagrangian submanifolds. We start by giving these definitions in the case where the symplectic manifold is $(\mathbb{C}^n, \omega_0)$. We give examples of monotone Lagrangian embeddings in $(\mathbb{C}^n, \omega_0)$ and state Polterovich's theorem which gives a restriction on the possible values of the Maslov number for monotone Lagrangian embeddings $L \to (\mathbb{C}^n, \omega_0)$ ([16]). Finally, we give a definition of monotonicity for general symplectic manifolds and give some examples. The definitions and results listed here are mostly taken from [2], [15] and [16].

In the previous section, we introduced the Lagrangian Grassmannian Λ_n and showed that it can be identified with $U(n)/O(n)$ (Lemma 3.9). Consider the determinant map $U(n) \to \mathbb{S}^1$. It gives $U(n)$ the structure of a $SU(n)$-fiber bundle over $\mathbb{S}^1$. This map does not factor through as a map $U(n)/O(n) \to \mathbb{S}^1$ but the map

$$\begin{aligned} det^2 : U(n)/O(n) &\to \mathbb{S}^1 \\ [A] &\to (detA)^2 \end{aligned}$$

is well defined. This is a fiber bundle with fiber $SU(n)/SO(n)$. Consider the homotopy long exact sequence of the fibration

$$SO(n) \to SU(n) \to SU(n)/SO(n)$$

Since $SU(n)$ is simply-connected and $SO(n)$ is connected, $SU(n)/SO(n)$ is simply-conneted. Finally, using the homotopy long exact sequence of the fibration

$$SU(n)/SO(n) \to U(n)/O(n) \xrightarrow{det^2} \mathbb{S}^1$$

we deduce that the det^2 map induces an isomorphism.

$$\pi_1(\Lambda_n) = \pi_1(U(n)/O(n) \xrightarrow{(det^2)_*} \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$$

Definition 4.1. The universal Maslov class is the generator of the cohomology group $H^1(\Lambda_n; \mathbb{Z}) = Hom(\pi_1(\Lambda_n), \mathbb{Z}) \simeq \mathbb{Z}$ given by

$$\mu_n = (det^2)^* \alpha$$

where α is the generator of $H^1(\mathbb{S}^1; \mathbb{Z})$.

Suppose $f : L \to (\mathbb{C}^n, \omega_0)$ is a Lagrangian immersion. As in the previous section, we can construct the Gauss map. The image of the map will be a subset of Λ_n since f is Lagrangian.

$$\begin{aligned} \tilde{f} &: L \to \Lambda_n \\ x &\to f_*(T_x L) \end{aligned}$$

Definition 4.2. Suppose $f : L \to (\mathbb{C}^n, \omega_0)$ is a Lagrangian immersion and $\tilde{f} : L \to \Lambda_n$ the corresponding Gauss map. The pullback $\tilde{f}^*\mu_n = \mu(f)$ is called the Maslov class of the immersion f. The Maslov number $N(f)$ is the non-negative generator of the subgroup $\mu(f)(H_1(L)) \subseteq \mathbb{Z}$.

In the case where $f : L \to (\mathbb{C}^n, \omega_0)$ is a Lagrangiam embedding, we will sometimes use the notations μ_L and N_L instead of $\mu(f)$ and $N(f)$ for the Maslov class and the Maslov number respectively. It can be shown that the Maslov class has the following additive property: If $f_1 : L \to (\mathbb{C}^{n_1}, \omega_0)$ and $f_2 : L \to (\mathbb{C}^{n_2}, \omega_0)$ are Lagrangian immersions, then the Maslov class of the induced Lagrangian immersion $f_1 \times f_2 : L_1 \times L_2 \to (\mathbb{C}^{n_1+n_2}, \omega_0)$ is given by $\mu(f) = \mu(f_1) + \mu(f_2) \in H^1(L_1 \times L_2; \mathbb{Z}) = H^1(L_1; \mathbb{Z}) \oplus H^1(L_2; \mathbb{Z})$. Also, note that the Maslov class of a Lagrangian immersion only depends on its l-regular homotopy class. Indeed, if $f, g : L \to (\mathbb{C}^n, \omega_0)$ are l-regularly homotopic Lagrangian immersions, the l-regular homotopy

$$F : L \times I \to (\mathbb{C}^n, \omega_0)$$

induces a homotopy

$$\tilde{F} : L \times I \to \Lambda_n$$

of the Gauss maps $\tilde{f}, \tilde{g} : L \to \Lambda_n$ and, hence, $\mu(f) = \mu(g)$.

Example 4.3. If we view $\mathbb{S}^1$ as the set $\{z \in \mathbb{C} |\ z\bar{z} = 1\}$, then the map

$$\begin{aligned} \mathbb{S}^1 &\to \mathbb{C} \\ z &\to z^n \end{aligned}$$

is a Lagrangian immersion for each $n \in \mathbb{Z}$. The Maslov class is

$$\begin{aligned} \mu(f) : H^1(\mathbb{S}^1) \simeq \mathbb{Z} &\to \mathbb{Z} \\ 1 &\to 2n \end{aligned}$$

Indeed, if we use the parametrisation $\mathbb{R} \to \mathbb{S}^1$, $t \to e^{2\pi i t}$, the Gauss map sends each $t \in \mathbb{R}$ to the real line spanned by the vector $(-sin(2\pi n t), cos(2\pi n t))$ in $\mathbb{R}^2$. The class $\mu(f)$ is given by $\tilde{f}^*\mu_n = (\tilde{f} \circ det^2)^* dt$. The composite $\tilde{f} \circ det^2$ is the map

$$\begin{array}{rcl} \mathbb{S}^1 & \to & \mathbb{S}^1 \\ e^{2\pi i t} & \to & -e^{4\pi i n t} = e^{2\pi i (2nt + \frac{1}{2})} \end{array}$$

Hence, we deduce that $\mu(f) = 2ndt$ and $m(f) = 2n$. Finally, given any embedding $\iota : \mathbb{S}^1 \to (\mathbb{C}, \omega_0)$, the Maslov number is 2. Indeed, the embedding defines a simple closed curve C in $\mathbb{R}^2$. Suppose $\gamma : [0,1] \to \mathbb{R}^2$ is a parametrization of the curve. Let $\theta(t)$ be the counterclockwise measured angle from the positive x-axis to the vector $\frac{d\gamma}{dt}$ at t. Then, in terms of the parametrizaton, the Gauss map is given by

$$\begin{array}{rcl} [0,1] & \to & \mathbb{S}^1 \simeq \Lambda_1 \\ t & \to & e^{i\theta(t)} \end{array}$$

The Maslov class $\mu(\iota)$ is defined by

$$\begin{array}{rcl} \mu(\iota)([\mathbb{S}^1]) & = & \frac{1}{\pi} \int \frac{d\theta}{dt} \\ \mu(\iota)([\mathbb{S}^1]) & = & \pm 2 \end{array}$$

The last equation follows from the Theorem of turning tangents (see [5]). The sign $\pm$ depends on the orientation of the curve. Therefore, the Maslov number is 2.

Example 4.4. If L is an oriented manifold of dimension n, then given any Lagrangian immersion $f : L \to (\mathbb{C}^n, \omega_0)$, the corresponding class $\mu(f)$ is even: it maps $H_1(L)$ to $2\mathbb{Z}$. This can be seen as follows. Consider $\tilde{\Lambda}_n$ the space of oriented Lagrangian subspaces of $V = (\mathbb{R}^{2n}, \omega_o)$. This set $\tilde{\Lambda}_n$ can be identified with $U(n)/SO(n)$. In fact, the determinant map induces a well-defined map

$$\begin{array}{r} det : U(n)/SO(n) \to \mathbb{S}^1 \\ [A] \to det A \end{array}$$

which gives $\tilde{\Lambda}_n$ the structure of a $SU(n)/SO(n)$-fiber bundle over $\mathbb{S}^1$. Just as in the case for Λ_n, we can use the homotopy long exact sequence of the fibration to show that we have an isomorphism

$$\pi_1(\tilde{\Lambda}_n) = \pi_1(U(n)/SO(n) \xrightarrow{(det^2)_*} \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$$

On the other hand, note that the map $\tilde{\Lambda}_n \xrightarrow{\phi} \Lambda_n$ which sends an oriented Lagrangian subspace of V to the underlying Lagrangian subspace gives a double cover of Λ_n. Furthermore, the Gauss map $\tilde{f}$ of f factors as

$$L \to \tilde{\Lambda}_n \xrightarrow{\phi} \Lambda_n.$$

Applying the H^1 functor, we see that the induced map $\tilde{f}^*$ factors as

$$H^1(\Lambda_n;\mathbb{Z}) \to H^1(\tilde{\Lambda_n};\mathbb{Z}) \to H^1(L;\mathbb{Z})$$

Hence, $\mu(f)$ is even since the homomorphism $H^1(\tilde{\Lambda_n};\mathbb{Z}) \to H^1(L;\mathbb{Z})$ simply corresponds to

$$\begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ 1 & \to & 2 \end{array}$$

We can now give the definition of monotone Lagrangian embeddings in $(\mathbb{C}^n, \omega_0)$. In the previous section, we defined the symplectic area class of a Lagrangian immersion $f : L \to (\mathbb{C}^n, \omega_0)$.

Definition 4.5. A Lagrangian immersion $f : L \to (\mathbb{C}^n, \omega_0)$ is said to be monotone if $\mu(f) = c\omega(f)$ for some real number $c > 0$.

If $f : L \to (\mathbb{C}^n, \omega_0)$ is a Lagrangian embedding, we will say that L is a monotone Lagrangian submanifold.

Example 4.6. The Whitney immersion $f : \mathbb{S}^n \to (\mathbb{C}^n, \omega_0)$ is trivially monotone since $\mu(f) = 0$ and $\omega(f) = 0$. For $n > 1$, this is due to the fact that $H^1(\mathbb{S}^n) = 0$. For $n = 1$, we get the result by doing a simple computation.

Example 4.7. The torus $T^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | \, |z| = r\}$ is a monotone Lagrangian submanifold of $\mathbb{C}^n$ for any $r > 0$: If we represent the torus as the set $\{(re^{2\pi i t_1}, \cdots, re^{2\pi i t_n}) \in \mathbb{C}^n | t_i \in \mathbb{R}\}$, then $H_1(T^n) = \mathbb{Z}^{\oplus n}$ is generated by the homology class of the loops

$$\begin{array}{rcl} \gamma_i : \mathbb{S}^1 & \to & L \\ e^{i2\pi t} & \to & (r, \cdots, \underbrace{re^{i2\pi t}}_{ith\ position}, \cdots, r) \end{array}$$

A simple computation gives us $\omega_{T^n}([\gamma_i]) = \int_0^{2\pi} r^2 \cos^2(\theta) d\theta = r^2\pi$ for each i. Based on the additive property of the Maslov class, we see that the computation of $\mu_{T^n}([\gamma_i])$ reduces to the computation of the Maslov number for the natural Lagrangian embedding $\mathbb{S}^1 \to \mathbb{C}$. We get $\mu_{T^n}([\gamma_i]) = 2$ for each i. Hence T^n is monotone with $c = \frac{2}{r^2\pi}$.

We see that the Maslov number for the natural Lagrangian embedding $T^n \to (\mathbb{C}^n, \omega_0)$ is 2. This turns out to be true for any Lagrangian embedding of the n-torus in $(\mathbb{C}^n, \omega_0)$.

Theorem 4.8. *[4] The Maslov number of a monotone Lagrangian torus in $(\mathbb{C}^n, \omega_0)$ is always 2.*

In general, the monotonicity condition imposes restrictions on the Maslov number of closed Lagrangian submanifolds $L \in (\mathbb{C}^n, \omega_0)$.

Theorem 4.9. *[16] If $f : L \to \mathbb{C}^n$ is a monotone Lagrangian embedding of a closed manifold L, then $1 \leq N_L \leq n+1$.*

In [16], Polterovich also gives examples of such Lagrangian submanifolds.

Lemma 4.10. *For every pair of integers $2 \leq k \leq n$, there exists a closed monotone Lagrangian submanifold L of $(\mathbb{C}^n, \omega_0)$ such that $N_L = k$.*

Proof. Consider the Lagrangian embedding $\iota : (\mathbb{S}^1 \times \mathbb{S}^{n-1})/\sim \to (\mathbb{C}^n, \omega_0)$ constructed in Example 1.53. We will denote this Lagrangian submanifold by L_n. We will show that it is monotone and $N_{L_n} = n$. We can view this manifold as a bundle over $\mathbb{S}^1$ with fiber $\mathbb{S}^{n-1}$. Indeed, let $I = [0,1]$ be the unit interval. Then, we have the bundle

$$L_n = (I \times \mathbb{S}^{n-1})/\sim \to I/\partial I = \mathbb{S}^1$$

where $(0,x) \sim (1,-x)$. Consider the case where $n \geq 3$. By using the homotopy long exact sequence of the fibration $\mathbb{S}^{n-1} \to L_n \to \mathbb{S}^1$, we get that

$$\pi_1(L_n) \simeq \pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$$

On the other hand, $\mathbb{S}^1 \times \mathbb{S}^{n-1} \to L_n = (\mathbb{S}^1 \times \mathbb{S}^{n-1})/\sim$ is a double covering map. Also note that the map

$$\begin{aligned} f : \mathbb{S}^1 \times \mathbb{S}^{n-1} &\to (\mathbb{C}^n, \omega_0) \\ (z,x) &\to zx \end{aligned}$$

is a Lagrangian immersion. This map factors as

$$\mathbb{S}^1 \times \mathbb{S}^{n-1} \to L_n \xrightarrow{\iota} (\mathbb{C}^n, \omega_0)$$

and, hence, the corresponding Gauss map $\tilde{f}$ factors as

$$\mathbb{S}^1 \times \mathbb{S}^{n-1} \to L_n \xrightarrow{\iota} \Lambda_n$$

Applying the H^1-functor, we get

$$H^1(\Lambda_n; \mathbb{Z}) \xrightarrow{\iota^*} H^1(L_n; \mathbb{Z}) \to H^1(\mathbb{S}^1 \times \mathbb{S}^{n-1}; \mathbb{Z})$$

Since $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ is a double cover of L_n, and the cohomolgy groups $H^1(L_n; \mathbb{Z})$ and $H^1(\mathbb{S}^1 \times \mathbb{S}^{n-1}; \mathbb{Z})$ are isomorphic to $\mathbb{Z}$, we see that

$$\begin{array}{rcl} H^1(L_n; \mathbb{Z}) & \to & H^1(\mathbb{S}^1 \times \mathbb{S}^{n-1}; \mathbb{Z}) \\ \alpha & \to & 2\theta \end{array}$$

where α and θ are generators of $H^1(L_n; \mathbb{Z})$ and $H^1(\mathbb{S}^1 \times \mathbb{S}^{n-1}; \mathbb{Z})$ respectively. This implies that $N(f) = 2N_L$. So, if we can compute the Maslov number $N(f)$, we can determine N_L. Also, $\omega(f)(\alpha) = 2\omega_{L_n}(\theta)$. Therefore, L_n is a monotone Lagrangian submanifold if and only if f is a monotone Lagrangian immersion.

We will now determine the Maslov number $N(f)$. As in the previous sections, we view the $(n-1)$-sphere as the set

$$\mathbb{S}^{n-1} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$$

Consider the open set $U = \{(x_1, \cdots, x_n) \in \mathbb{S}^{n-1} \, | x_n > 0\}$. If we use the parametrisation

$$(t; (x_1, \cdots, x_{n-1})) \to \left(e^{i2\pi t}; (x_1, \cdots, x_{n-1}, \sqrt{1 - |x|^2})\right)$$

on the open subset $\mathbb{S}^1 \times U$ of $\mathbb{S}^1 \times \mathbb{S}^n$, then the map $f : \mathbb{S}^1 \times U \to \mathbb{C}^n$ is given by

$$(t; (x_1, \cdots, x_{n-1})) \to (x_1 e^{i2\pi t}, \cdots, x_{n-1} e^{i2\pi t}, \sqrt{1 - |x|^2} e^{i2\pi t})$$

Furthermore, for each $p = (t; (x_1, \cdots, x_{n-1}))$, the corresponding Lagrangian subspace $T_{f(p)}L \subset (\mathbb{C}^n, \omega_0)$ is the real subspace spanned by the column vectors of the Jacobian matrix of the above map. Hence, after doing a simple computation, we see that the Gauss map $\tilde{f}$ sends each $(t; (x_1, \cdots, x_{n-1}))$ to the real subspace spanned by the vectors

$$\begin{pmatrix} ie^{i2\pi t}x_1 \\ \vdots \\ \vdots \\ ie^{i2\pi t}x_{n-1} \\ ie^{i2\pi t}\sqrt{1-|x|^2} \end{pmatrix}, \begin{pmatrix} e^{i2\pi t} \\ 0 \\ \vdots \\ \vdots \\ \frac{-e^{i2\pi t}x_1}{\sqrt{1-|x|^2}} \end{pmatrix}, \begin{pmatrix} 0 \\ e^{i2\pi t} \\ 0 \\ \vdots \\ \frac{-e^{i2\pi t}x_2}{\sqrt{1-|x|^2}} \end{pmatrix}, \cdots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{i2\pi t} \\ \frac{-e^{i2\pi t}x_{n-1}}{\sqrt{1-|x|^2}} \end{pmatrix}$$

in $\mathbb{C}^n$. The homolgy group $H_1(\mathbb{S}^1 \times \mathbb{S}^{n-1})$ is generated by the homology class of the loop

$$\begin{aligned} \gamma : \mathbb{S}^1 &\to \mathbb{S}^1 \times \mathbb{S}^{n-1} \\ z &\to (z, (0, \cdots, 1)) \end{aligned}$$

If we use our parametrisation on $\mathbb{S}^1 \times U$, we see that the corresponding loop $\tilde{f} \circ \gamma : [0,1] \to \Lambda_n$ maps each $t \in [0,1]$ to the real vector space spanned by the vectors

$$\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ ie^{i2\pi t} \end{pmatrix}, \begin{pmatrix} e^{i2\pi t} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^{i2\pi t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{i2\pi t} \\ 0 \end{pmatrix}$$

Note that, for each $t \in L$, the above vectors give a unitary basis for $\mathbb{C}^n$. If we recall our identification $\Lambda_n = U(n)/O(n)$, we deduce that the map $\tilde{f} \circ \gamma$ sends each $t \in [0,1]$ to the equivalence class of the diagonal matrix

$$\begin{pmatrix} e^{i2\pi t} & 0 & \dots & 0 & 0 \\ 0 & e^{i2\pi t} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e^{i2\pi t} & 0 \\ 0 & \dots & \dots & 0 & ie^{i2\pi t} \end{pmatrix}$$

Finally, applying the map $\det^2 : U(n)/O(n) \to \mathbb{S}^1$, we get the loop

$$\begin{aligned} \mathbb{S}^1 &\to \mathbb{S}^1 \\ e^{i2\pi t} &\to e^{i4\pi nt + i\pi} \end{aligned}$$

Since $\mu(f)(\gamma)$ is the degree of this map, we deduce that the Maslov number of f is $N(f) = 2n$. On the other hand, a simple computation gives us $\omega(f)([\gamma]) = \pi$. Hence, L_n is a monotone Lagrangian submanifold of $(\mathbb{C}^n, \omega_0)$ and $N_{L_n} = n$.

By using arguments similar to the ones used above, one can show that, for $n = 2$, $L_2 \to (\mathbb{C}^2, \omega_0)$ is a monotone Lagrangian embedding with $N_L = 2$

Now, suppose $2 \leq k < n$ and consider the Lagrangian embedding

$$\iota : L_k \to (\mathbb{C}^k, \omega_0)$$

discussed above and the Whitney immersion

$$f : \mathbb{S}^{n-k} \to (\mathbb{C}^{n-k}, \omega_0).$$

We showed that this immersion is monotone with a minimal Maslov number equal to zero. Then, the map

$$\iota \times f : L_k \times \mathbb{S}^{n-k} \to (\mathbb{C}^n, \omega_0)$$

is a Lagrangian immersion. By construction, it is monotone and, based on the additive property of the Maslov class, the Maslov number is equal to k. Since L_k, $\mathbb{S}^{n-k}$ are closed manifolds and ι is an embedding, we can apply Lemma 2.16. There is a Lagrangian embedding $j : L_k \times \mathbb{S}^{n-k} \to (\mathbb{C}^n, \omega_0)$ which is l-regular homotopic to $\iota \times f$. Hence, this Lagrangian embedding is monotone and its minimal Maslov number is equal to k. This completes the proof. □

We get the followinng result from Theorem 4.9.

Corollary 4.11. *Suppose $f : L \to (\mathbb{C}^n, \omega_0)$ is a Lagrangian immersion with $N(f) = 0$ or $N \geq (n+1)$. Then f is not l-regular homotopic to a Lagrangian embedding $L \to (\mathbb{C}^n, \omega_0)$.*

The definition of monotonicity that we have given for Lagrangian submmanifolds in $(\mathbb{C}^n, \omega_0)$ is based on the following two properties of this symplectic manifold.

1. The symplectic vector bundle $(T\mathbb{C}^n, \omega_0) \to \mathbb{C}^n$ is trivial.

2. ω_0 is an exact form.

These properties hold for any symplectic structure ω on $\mathbb{C}^n$. In fact, we can, in a similar way, define the Maslov class and the symplectic area class of Lagrangian immersions or embeddings $L \to (\mathbb{C}^n, \omega)$. On the other hand, general symplectic manifolds (M, ω) do not satisfy the properties listed above. For instance, the symplectic form ω cannot be exact if M is a closed manifold. So, we will need a more general definition of monotone Lagrangian submanifolds.

Suppose (M, ω) is a symplectic manifold. By using the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$, we see that the cohomology class $[\omega] \in H^2(M; \mathbb{R})$ of the sympletic form and the first Chern class $c_1(TM) \in H^2(M; \mathbb{Z})$ define homomorphisms

$$\omega : \pi_2(M) \to \mathbb{R} \text{ and } c_1 : \pi_2(M) \to \mathbb{R}$$

respectively. If $\alpha \in \pi_2(M)$ is represented by a map $h : \mathbb{S}^2 \to M$, then the first homorphism is given by

$$\omega(\alpha) = \int_{\mathbb{S}^2} h^* \omega.$$

Definition 4.12. A sympletctic manifold (M, ω) is monotone if there exists $b > 0$ such that

$$c_1(\alpha) = b\omega(\alpha)$$

for any $\alpha \in \pi_2(M)$.

Example 4.13. Consider the manifold $\mathbb{C}P^n$. The chern class $c_1(T\mathbb{C}P^n)$ is equal to $(n+1)\nu$ where ν is the generator of $H^2(\mathbb{C}P^n) \simeq \mathbb{Z}$ defined by $\nu([\mathbb{C}P^1]) = 1$. Here, $[\mathbb{C}P^1] \in H_2(\mathbb{C}P^n) \simeq \pi_2(\mathbb{C}P^n) \simeq \mathbb{Z}$ is the homology class of a complex projective line $\mathbb{C}P^1 \subset \mathbb{C}P^n$. Given any symplectic form ω on $\mathbb{C}P^n$, $\nu = \frac{1}{a}[\omega]$ where $a = \omega([\mathbb{C}P^1])$. Hence, $(\mathbb{C}P^n, \omega)$ is monotone with $b = \frac{n+1}{a}$. For the symplectic structure defined in Example 1.32, a simple computation gives $b = \frac{n+1}{\pi}$.

Definition 4.14 (The Maslov index)**.** Suppose (M, ω) is a symplectic manifold and L a Lagrangian submanifold. The Maslov index is a homomorphism

$$\mu_L : \pi_2(M, L) \to \mathbb{Z}$$

constructed as follows. Suppose $\beta \in \pi_2(M, L)$ is represented by a map

$$f : (\mathbb{D}^2, \mathbb{S}^1) \to (M, L)$$

We will denote by $(E_f, \omega) \to \mathbb{D}^2$ the pullback symplectic vector bundle $(f^*TM, \omega) \to \mathbb{D}^2$. Then, the pullback bundle $f^*TL \to \mathbb{S}^1$ is a Lagrangian subbundle of $(E_f|_{\mathbb{S}^1}, \omega) \to \mathbb{S}^1$. Since $\mathbb{D}^2$ is a contractible space, there is a trivialization

$$\begin{array}{ccc} (E_f, \omega) & \xrightarrow{\cong} & \mathbb{D}^2 \times (\mathbb{C}^n, \omega_0) \\ \downarrow & & \downarrow \\ \mathbb{D}^2 & = & \mathbb{D}^2 \end{array}$$

which is unique up to homotopy. Using this trivialization and the Lagrangian subbundle $f^*TL \to \mathbb{S}^1$ of $(E_f|_{\mathbb{S}^1}, \omega) \to \mathbb{S}^1$, we get a map $\gamma_f : \mathbb{S}^1 \to \Lambda_n$ and, hence, an element $[\gamma_f] \in H_1(\Lambda_n)$. We set

$$\mu_L(\beta) = \mu([\gamma_f]) \tag{4.1}$$

where $\mu \in H^1(\Lambda_n)$ is the universal Maslov class. The minimal Maslov number N_L is the non-negative generator of the subgroup $\mu_L(\pi_2(M, L)) \subset \mathbb{Z}$.

In the case where $(M, \omega) = (\mathbb{C}^n, \omega_0)$, it can be shown that this definition agrees with the original defnition. The Maslov index μ_L satisfies the following important property.

Lemma 4.15. *Suppose L is a Lagrangian submanifold of the symplectic manifold (M, ω) and $f_1, f_2 : (\mathbb{D}^2, \mathbb{S}^1) \to (M, L)$ are maps such that $f_1|_{\mathbb{S}^1} = f_2|_{\mathbb{S}^1}$. Let $f_1 \# f_2 : \mathbb{S}^2 \to M$ be the map defined by*

$$f_1 \# f_2(z) = \left\{ \begin{array}{ll} f_1(z) & z \in \mathbb{D}^2 \\ f_2(z) & z \in \overline{\mathbb{D}}^2 \end{array} \right\}$$

where $\overline{\mathbb{D}}^2$ denotes the 2-disc with the opposite orientation. Then

$$\mu_L([f_1]) - \mu_L([f_2]) = 2c_1([f_1 \# f_2]).$$

The symplectic form ω also defines a homomorphism $\omega : \pi_2(M, L) \to \mathbb{R}$ given by

$$\omega([f]) = \int_{\mathbb{D}^2} f^*\omega$$

where $[f] \in \pi_2(M, L)$. This homomorphism satisfies

$$\omega([f_1 \# f_2]) = \omega([f_1]) - \omega([f_2]).$$

Definition 4.16. A Lagrangian submanifold L of a closed symplectic manifold (M, ω) is monotone if there exists $c > 0$ such that

$$\mu_L(\beta) = c\omega(\beta)$$

for all $\beta \in \pi_2(M, L)$.

If a closed connected symplectic manifold (M, ω) admits a monotone Lagangian submanifold L, then (M, ω) must be monotone: Suppose the homotopy class $\alpha \in \pi_2(M)$ is represented by a map $h : (\mathbb{S}^2, e_0) \to (M, x_0)$ where e_0 and $x_0 \in L$ are basepoints of $\mathbb{S}^2$ and M respectively. This corresponds to a map $f : (\mathbb{D}^2, \mathbb{S}^1) \to (M, x_0)$. In fact, we can express h as $f \# c_{x_0}$ where $c_{x_0} : \mathbb{D}^2 \to M$ is the constant map. Using Lemma 4.15, we get

$$\begin{aligned} \mu_L([f]) - \mu_L([c_{x_0}]) &= 2c_1([f_1 \# c_{x_0}]) \\ \Rightarrow \mu_L([f]) &= 2c_1(\alpha) \end{aligned}$$

Since L is monotone, the last equation gives us

$$\begin{aligned} c\omega([f]) &= 2c_1(\alpha) \\ \Rightarrow c\omega(\alpha) &= 2c_1(\alpha) \end{aligned}$$

Hence (M, ω) is monotone with $2b = c$.

Example 4.17. Suppose (M, ω) is a closed monotone symplectic manifold and L is a connected Lagrangian submanifold L with $\pi_1(L)$ torsion. Then L is monotone: Suppose the class $\beta \in \pi_2(M, L)$ is represented by a map $f : (\mathbb{D}^2, \mathbb{S}^1) \to (M, L)$. Since $\pi_1(L)$ is torsion, there is a nonnegative integer r such that $[f|_{\mathbb{S}^1}]^r = 0 \in \pi_1(L)$. Hence, if $\beta^r \in \pi_2(M, L)$ is represented by a map $q : (\mathbb{D}^2, \mathbb{S}^1) \to (M, L)$, there is a map $k : \mathbb{D}^2 \to L$ such that $q|_{\mathbb{S}^1} = k|_{\mathbb{S}^1}$. Then $q \# k : \mathbb{S}^2 \to M$ and, by Lemma 4.15, we get

$$\mu_L([q]) - \mu_L([k]) = 2c_1([q \# k]).$$

Since (M, ω) is monotone, this equation gives us

$$\begin{aligned} \mu_L(\beta^r) - \mu_L([k]) &= 2b\omega([q \# k]) \\ \Rightarrow \mu_L(\beta^r) - \mu_L([k]) &= 2b\left(\omega(\beta^r) - \omega([k])\right) \\ \Rightarrow r\mu_L(\beta) - \mu_L([k]) &= 2b\left(r\omega(\beta) - \omega([k])\right). \end{aligned}$$

Note that $\omega([k]) = 0$ since the image of k lies in L and $\omega|_L = 0$. Also, $\mu_L([k]) = 0$. Indeed, the corresponding loop $\gamma_k : \mathbb{S}^1 \to \Lambda_n$ in Equation 4.1 extends to a map $\mathbb{D}^2 \to \Lambda_n$. This implies that γ_k is contractible and $\mu_L([k]) = \mu([\gamma_k]) = 0$. So we are left with $\mu_L(\beta) = 2b\omega(\beta)$. This implies that L is monotone.

Example 4.18. Consider the symplectic manifold $(\mathbb{S}^2, \omega)$ where ω is the area form from Example 1.29. The first Chern class $c_1(T\mathbb{S}^2)$ is given by $c_1 = \frac{1}{2\pi}[\omega]$. Any embedding $\iota : \mathbb{S}^1 \to \mathbb{S}^2$ is Lagrangian. Furthermore,

$$\pi_2(\mathbb{S}^2, \mathbb{S}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

where the summands are generated by the homotopy classes of the maps

$$f_1, f_2 : \mathbb{D}^2 \to \mathbb{S}^2$$

which represent the two discs having the Lagrangian $\mathbb{S}^1$ as a common boundary. We have $[f_1 \# f_2] = \pm[\mathbb{S}^2]$ where $[\mathbb{S}^2]$ is the fundamental class of the 2-sphere. Without loss of generality, assume $[f_1 \# f_2] = [\mathbb{S}^2]$. The Maslov index can be computed as follows. Let $U \to \mathbb{C}$ be a stereographic projection map where U is an open set containing $f_1(\mathbb{D}^2)$. Using this map, we can embed the Lagrangian $\mathbb{S}^1$ of $(\mathbb{S}^2, \omega)$ as a Lagrangian submanifold of $(\mathbb{C}, \omega_0)$. Then computing $\mu_{\mathbb{S}^1}([f_1])$ reduces to computing the Maslov number for the constructed Lagrangian embedding $\mathbb{S}^1 \to (\mathbb{C}, \omega_0)$. We get $\mu_{\mathbb{S}^1}([f_1]) = 2$. Now, if we use Lemma 4.15, we get

$$\begin{aligned} \mu_{\mathbb{S}^1}([f_1]) - \mu_{\mathbb{S}^1}([f_2]) = 2c_1([\mathbb{S}^2]) = 4 \\ \Rightarrow \mu_{\mathbb{S}^1}([f_2]) = -2. \end{aligned}$$

Hence, the map $\iota : \mathbb{S}^1 \to \mathbb{S}^2$ is a monotone Lagrangian embedding if and only if

$$\omega([f_1]) = 2\pi \tag{4.2}$$

$$\omega([f_2]) = -2\pi. \tag{4.3}$$

In other words, the circle $\mathbb{S}^1$ divides the area of the 2-sphere $\mathbb{S}^2$ in half.

Lemma 4.19. *Suppose (M, ω) is a closed monotone symplectic manifold and suppose $\Phi : M \to M$ is an anti-symplectic involution : $\Phi^*\omega = -\omega$ and $\Phi \circ \Phi = Id$. Let $L = \{x \in M | \Phi(x) = x\}$ be the set of fixed points of Φ. If L is nonempty, then L is a closed monotone Lagrangian submanifold of M.*

Proof. We begin by proving that L is a closed submanifold of M with dimension $\dim L = \frac{\dim M}{2}$. The map Φ defines a $\mathbb{Z}/2\mathbb{Z}$ group action on M. Recall that the set of fixed points of a G-group action on a closed manifold is a closed submanifold ([3]). Hence, L is a closed submanifold of M. For any $x \in L$, the differential $d_x\Phi$ is a linear isomorphism $T_xM \to T_xM$. Since $d_x\Phi \circ d_x\Phi = Id$, the eigenvalues of $d_x\Phi$ are 1 and -1 with eigenspaces

$$\begin{aligned} V_1 &= \{u + d_x\Phi(u) |\, u \in T_xM\} = T_xL \\ V_2 &= \{u - d_x\Phi(u) |\, u \in T_xM\}. \end{aligned}$$

Furthermore, we have the isomorphism

$$\begin{aligned} V_1 &\to V_2 \\ u + d_x\Phi(u) &\to u - d_x\Phi(u) \end{aligned}$$

and the decomposition

$$\begin{aligned} T_xM &\to V_1 \oplus V_2 \\ u &\to \frac{1}{2}(u + d_x\Phi(u)) + \frac{1}{2}(u - d_x\Phi(u)). \end{aligned}$$

Hence, we get $\dim L = \frac{\dim M}{2}$ as claimed. Furthermore, $\omega|_L = 0$. Indeed, given $x \in L$ and $u, v \in T_xL$, we have

$$\begin{aligned} \omega_x(u,v) &= -(\Phi^*\omega)_x(u,v) \\ \Rightarrow \omega_x(u,v) &= -\omega_x(d_x\Phi(u), d_x\Phi(v)) \\ \Rightarrow \omega_x(u,v) &= -\omega_x(u,v) \\ \Rightarrow \omega_x(u,v) &= 0 \end{aligned}$$

Hence, L is a Lagragian submanifold of (M, ω). Finally, we will show that L is monotone. Suppose $[f] \in \pi_2(M, L)$. Then $\Phi \circ f : (\mathbb{D}^2, \mathbb{S}^1) \to (M, L)$ and $f|_{\mathbb{S}^1} = \Phi \circ f|_{\mathbb{S}^1}$ since L is the fixed point set of Φ. We can construct the map $h = f\#(\Phi \circ f) : \mathbb{S}^2 \to M$. Using Lemma 4.15, we get

$$\mu_L([f]) - \mu_L([\Phi \circ f]) = 2c_1([h])$$

Since M is monotone, we deduce that

$$\begin{aligned} \mu_L([f]) - \mu_L([\Phi \circ f]) &= 2b\omega([h]) \\ \Rightarrow \mu_L([f]) - \mu_L([\Phi \circ f]) &= 2b\left(\omega([f]) - \omega([\Phi \circ f])\right). \end{aligned} \tag{4.4}$$

Note that

$$\omega([\Phi \circ f]) = \int_{\mathbb{D}^2} (\Phi \circ f)^*\omega = \int_{\mathbb{D}^2} f^*(\Phi^*\omega) = \int_{\mathbb{D}^2} f^*(-\omega) = -\omega([f])$$

Also $\mu_L([\Phi \circ f]) = -\mu_L([f])$: We have a symplectic vector bundle isomorphism

$$\begin{array}{ccc} (f^*TM, -\omega) & \to & ((\Phi \circ f)^*TM, \omega) \\ \downarrow & & \downarrow \\ \mathbb{D}^2 & = & \mathbb{D}^2 \end{array}$$

given by

$$(x, u) \to (x, d_{f(x)}\Phi(u))$$

where $x \in \mathbb{D}^2$ and $u \in T_{f(x)}M$. Hence, a trivialization

$$(f^*TM, \omega) \simeq \mathbb{D}^2 \times (\mathbb{C}^n, \omega_0)$$

induces a trivialization

$$((\Phi \circ f)^*TM, \omega) \simeq (f^*TM, -\omega) \simeq \mathbb{D}^2 \times (\mathbb{C}^n, -\omega_0)$$

This implies that the corresponding loop $\gamma_{\Phi\circ f} : \mathbb{S}^1 \to \Lambda_n$ needed in Equation 4.1 is the reverse of the loop γ_f. So, we do get $\mu_L([\Phi \circ f]) = -\mu_L([f])$. Equation 4.4 reduces to $\mu_L([f]) = 2b\omega([f])$. Thus, L is monotone. □

Example 4.20. Consider the symplectic manifold $(\mathbb{C}P^n, \omega)$ from Example 1.32. We showed in Example 4.13 that this is a closed monotone symplectic manifold. The map

$$\begin{array}{rcl} \Phi : \mathbb{C}P^n & \to & \mathbb{C}P^n \\ [z_0 : \cdots : z_n] & \to & [\overline{z_0} : \cdots : \overline{z_n}] \end{array}$$

is an anti-symplectic involution. Therefore, the fixed point set, $\mathbb{R}P^n$, is a closed monotone Lagrangian submanifold.

As we mentioned in the beginning of this section, monotonicity is a condition which is often imposed on symplectic manifolds and Lagrangian submanifolds in order to apply advanced machinery such as Floer (co)homology. The Floer (co)homology of a monotone symplectic manifold is the (co)homology of a chain complex constructed from a moduli space of Hamiltonianly perturbed pseudo-holomorphic cylinders (see [17]). The construction of this chain complex for general symplectic manifolds is more involved due to the existence of pseudoholomorphic spheres with negative Chern number. The monotonicity condition prevents this from happening.(It can be shown that $\omega([h]) > 0$ for any non-constant pseudo-holomorphic sphere on $h : \mathbb{S}^2 \to (M, \omega)$. Hence, if M is monotone, we get $c_1([h]) = c\omega([h]) > 0$.) The Floer (co)homology of a monotone symplectic manifold is isomorphic to the singular (co)homology of the manifold. There is also a version of Floer (co)homology for monotone Lagrangian submanifolds. It involves studying pseudoholomorphic discs with boundary lying on the Lagrangian submanifolds. By using this machinery, symplectic topologists can derive topological properties of monotone sympletic manifolds and monotone Lagrangian embeddings. The following Theorem of Albers is an example of such results.

Definition 4.21. A Lagrangian submanifold $L \in (M, \omega)$ is said to be Hamiltonianly displaceable if there exists a Hamiltonian vector field X which generates a flow of symplectomorphisms $\{\phi\}_{t\in\mathbb{R}}$ such that $\phi_1(L) \cap L = \emptyset$.

Theorem 4.22. *[1] Suppose (M, ω) is a monotone closed symplectic manifold and $\iota : L \to (M, \omega)$ a monotone Lagrangian embedding of a closed manifold with $N_L \geq 2$. If the Lagrangian submanifold is Hamitonianly diplaceable then the homomorphism*

$$\iota_k : H_k(L, \mathbb{Z}/2) \to H_k(M, \mathbb{Z}/2)$$

induced by the map ι is zero for $k > \dim L + 1 - N_L$.

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